A Survey on Solution Methods for Integral Equations*

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1 Introduction

Integral Equations arise naturally in applications, in many areas of Mathematics, Science and Technology and have been studied extensively both at the theoretical and practical level. It is noteworthy that a MathSciNet keyword search on Integral Equations returns more than eleven thousand items. In this survey we plan to describe several solution methods for Integral Equations, illustrated with a number of fully worked out examples. In addition, we provide a bibliography, for the reader who would be interested in learning more about various theoretical and computational aspects of Integral Equations.

Our interest in Integral Equations stems from the fact that understanding and implementing solution methods for Integral Equations is of vital importance in designing efficient parameterization algorithms for algebraic curves, surfaces and hypersurfaces. This is because the implicitization algorithm for algebraic curves, surfaces and hypersurfaces based on a nullvector computation, can be inverted to yield a parameterization algorithm.

Integral Equations are inextricably related with other areas of Mathematics, such as Integral Transforms, Functional Analysis and so forth. In view of this fact, and because we made a conscious effort to limit the length in under 50 pages, it was inevitable that the present work is not self-contained.

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2 Linear Integral Equations

2.1 General Form

The most general form of a linear integral equation is

$$h(x)u(x) = f(x) + \int_a^{b(x)} K(x,t)u(t)\,dt \tag{1}$$

The type of an integral equation can be determined via the following conditions:

$f(x) = 0$ Homogeneous...
$f(x) \neq 0$ Nonhomogeneous...
$b(x) = x$ ...Volterra integral equation...
$b(x) = b$ ...Fredholm integral equation...
$h(x) = 0$ ...of the 1st kind.
$h(x) = 1$ ...of the 2nd kind.

For example, Abel's problem:

$$-\sqrt{2g}f(x) = \int_0^x \frac{\phi(t)}{\sqrt{x-t}}\,dt \tag{2}$$

is a nonhomogeneous Volterra equation of the 1st kind.

2.2 Linearity of Solutions

If $u_1(x)$ and $u_2(x)$ are both solutions to the integral equation, then $c_1u_1(x) + c_2u_2(x)$ is also a solution.

2.3 The Kernel

$K(x,t)$ is called the kernel of the integral equation. The equation is called singular if:

- the range of integration is infinite
- the kernel becomes infinite in the range of integration

2.3.1 Difference Kernels

If $K(x,t) = K(x-t)$ (i.e. the kernel depends only on $x-t$), then the kernel is called a difference kernel. Volterra equations with difference kernels may be solved using either Laplace or Fourier transforms, depending on the limits of integration.

2.3.2 Resolvent Kernels

2.4 Relations to Differential Equations

Differential equations and integral equations are equivalent and may be converted back and forth:

$$\frac{dy}{dx} = F(x) \Rightarrow y(x) = \int_a^x \int_a^s F(t)\,dt\,ds + c_1 x + c_2$$

2.4.1 Reduction to One Integration

In most cases, the resulting integral may be reduced to an integration in one variable:

$$\int_a^x \int_a^s F(t)\,dt\,ds = \int_a^x F(t) \int_t^x ds\,dt = \int_a^x (x-t)F(t)\,dt$$

Or, in general:

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} F(x_n)\,dx_n\,dx_{n-1}\,\dots\,dx_1 = \frac{1}{(n-1)!}\int_a^x (x-x_1)^{n-1}F(x)\,dx_1$$

2.4.2 Generalized Leibnitz formula

The Leibnitz formula may be useful in converting and solving some types of equations.

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} F(x,y)\,dy = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x}(x,y)\,dy + F(x,\beta(x))\frac{d\beta}{dx}(x) - F(x,\alpha(x))\frac{d\alpha}{dx}(x)$$

3 Transforms

3.1 Laplace Transform

The Laplace transform is an integral transform of a function $f(x)$ defined on $(0,\infty)$. The general formula is:

$$F(s) \equiv \mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\,dx \tag{3}$$

The Laplace transform happens to be a Fredholm integral equation of the 1st kind with kernel $K(s,x) = e^{-sx}$.

3.1.1 Inverse

The inverse Laplace transform involves complex integration, so tables of transform pairs are normally used to find both the Laplace transform of a function and its inverse.

3.1.2 Convolution Product

When $F_1(s) \equiv \mathcal{L}\{f_1\}$ and $F_2(s) \equiv \mathcal{L}\{f_2\}$ then the convolution product is $\mathcal{L}\{f_1 * f_2\} \equiv F_1(s)F_2(s)$, where $f_1 * f_2 = \int_0^x f_1(x-t)f_2(t)\,dt$. $f_1(x-t)$ is a difference kernel and $f_2(t)$ is a solution to the integral equation. Volterra integral equations with difference kernels where the integration is performed on the interval $(0,\infty)$ may be solved using this method.

3.1.3 Commutativity

The Laplace transform is commutative. That is:

$$f_1 * f_2 = \int_0^x f_1(x-t)f_2(t)\,dt = \int_0^x f_2(x-t)f_1(t)\,dt = f_2 * f_1$$

3.1.4 Example

Find the inverse Laplace transform of $F(s) = \frac{1}{s^2+9} + \frac{1}{s(s+1)}$ using a table of transform pairs:

$$\begin{aligned}
f(x) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+9} + \frac{1}{s(s+1)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} \\
&= \frac{1}{3}\sin 3x + \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\
&= \frac{1}{3}\sin 3x + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\
&= \frac{1}{3}\sin 3x + 1 - e^{-x}
\end{aligned}$$

3.2 Fourier Transform

The Fourier exponential transform is an integral transform of a function $f(x)$ defined on $(-\infty, \infty)$. The general formula is:

$$F(\lambda) \equiv \mathcal{F}\{f\} = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx \tag{4}$$

The Fourier transform happens to be a Fredholm equation of the 1st kind with kernel $K(\lambda, x) = e^{-i\lambda x}$.

3.2.1 Inverse

The inverse Fourier transform is given by:

$$f(x) \equiv \mathcal{F}^{-1}\{F\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} F(\lambda)\, d\lambda \tag{5}$$

It is sometimes difficult to determine the inverse, so tables of transform pairs are normally used to find both the Fourier transform of a function and its inverse.

3.2.2 Convolution Product

When $F_1(\lambda) \equiv \mathcal{F}\{f_1\}$ and $F_2(\lambda) \equiv \mathcal{F}\{f_2\}$ then the convolution product is $\mathcal{F}\{f_1 * f_2\} \equiv F_1(\lambda)F_2(\lambda)$, where $f_1 * f_2 = \int_{-\infty}^{\infty} f_1(x-t)f_2(t)\, dt$. $f_1(x-t)$ is a difference kernel and $f_2(t)$ is a solution to the integral equation. Volterra integral equations with difference kernels where the integration is performed on the interval $(-\infty, \infty)$ may be solved using this method.

3.2.3 Commutativity

The Fourier transform is commutative. That is:

$$f_1 * f_2 = \int_{-\infty}^{\infty} f_1(x-t)f_2(t)\, dt = \int_{-\infty}^{\infty} f_2(x-t)f_1(t)\, dt = f_2 * f_1$$

3.2.4 Example

Find the Fourier transform of $g(x) = \frac{\sin ax}{x}$ using a table of integral transforms:

The *inverse* transform of $\frac{\sin a\lambda}{\lambda}$ is

$$f(x) = \begin{cases} \frac{1}{2}, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

but since the Fourier transform is symmetrical, we know that

$$\mathcal{F}\{F(x)\} = 2\pi f(-\lambda).$$

Therefore, let

$$F(x) = \frac{\sin ax}{x}$$

and

$$f(\lambda) = \begin{cases} \frac{1}{2}, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

so that

$$\mathcal{F}\left\{\frac{\sin ax}{x}\right\} = 2\pi f(-\lambda) = \begin{cases} \pi, & |\lambda| \leq a \\ 0, & |\lambda| > a \end{cases}.$$

3.3 Mellin Transform

The Mellin transform is an integral transform of a function $f(x)$ defined on $(0, \infty)$. The general formula is:

$$F(\lambda) \equiv \mathcal{M}\{f\} = \int_0^\infty x^{\lambda-1} f(x)\, dx \tag{6}$$

If $x = e^{-t}$ and $f(x) \equiv 0$ for $x < 0$ then the Mellin transform is reduced to a Laplace transform:

$$\int_0^\infty (e^{-t})^{\lambda-1} f(e^{-t})(-e^{-t})\, dt = -\int_0^\infty e^{-\lambda t} f(e^{-t})\, dt.$$

3.3.1 Inverse

Like the inverse Laplace transform, the inverse Mellin transform involves complex integration, so tables of transform pairs are normally used to find both the Mellin transform of a function and its inverse.

3.3.2 Convolution Product

When $F_1(\lambda) \equiv \mathcal{M}\{f_1\}$ and $F_2(\lambda) \equiv \mathcal{M}\{f_2\}$ then the convolution product is $\mathcal{M}\{f_1 * f_2\} \equiv F_1(\lambda)F_2(\lambda)$, where $f_1 * f_2 = \int_0^\infty \frac{f_1(t) f_2(\frac{x}{t})}{t}\, dt$. Volterra integral equations with kernels of the form $K(\frac{x}{t})$ where the integration is performed on the interval $(0, \infty)$ may be solved using this method.

3.3.3 Example

Find the Mellin transform of $e^{-ax}, a > 0$. By definition:

$$F(\lambda) = \int_0^\infty x^{\lambda-1} e^{-ax}\, dx.$$

Let $ax = z$, so that:

$$F(\lambda) = a^{-\lambda} \int_0^\infty e^{-z} z^{\lambda-1}\, dz.$$

From the definition of the gamma function $\Gamma(\nu)$

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x}\, dx$$

the result is

$$F(\lambda) = \frac{\Gamma(\lambda)}{a^\lambda}.$$

4 Numerical Integration

General Equation:

$$u(x) = f(x) + \int_a^b K(x,t)u(t)\, dt$$

Let $S_n(x) = \sum_{k=0}^n K(x,t_k)u(t_k)\Delta_k t$.
Let $u(x_i) = f(x_i) + \sum_{k=0}^n K(x_i,t_k)u(t_k)\Delta_k t, i = 0,1,2,\ldots,n$.
If equal increments are used, $\therefore \Delta x = \frac{b-a}{n}$.

4.1 Midpoint Rule

$$\int_a^b f(x)\, dx \approx \frac{b-a}{n} \sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right)$$

4.2 Trapezoid Rule

$$\int_a^b f(x)\, dx \approx \frac{b-a}{n} \left[\frac{1}{2} f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} f(x_n)\right]$$

4.3 Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

5 Volterra Equations

Volterra integral equations are a type of linear integral equation (1) where $b(x) = x$:

$$h(x)u(x) = f(x) + \int_a^x K(x,t)u(t)\,dt \tag{7}$$

The equation is said to be the first kind when $h(x) = 0$,

$$-f(x) = \int_a^x K(x,t)u(t)\,dt \tag{8}$$

and the second kind when $h(x) = 1$,

$$u(x) = f(x) + \int_a^x K(x,t)u(t)\,dt \tag{9}$$

5.1 Relation to Initial Value Problems

A general initial value problem is given by:

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y(x) = g(x) \tag{10}$$

$$y(a) = c_1,\ y'(a) = c_2$$

$$\therefore y(x) = f(x) + \int_a^x K(x,t)y(t)\,dt$$

where

$$f(x) = \int_a^x (x-t)g(t)\,dt + (x-a)[c_1A(a) + c_2] + c_1$$

and

$$K(x,t) = (t-x)[B(t) - A'(t)] - A(t)$$

5.2 Solving 2nd-Kind Volterra Equations

5.2.1 Neumann Series

The resolvent kernel for this method is:

$$\Gamma(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$$

where

$$K_{n+1}(x,t) = \int_t^x K(x,y)K_n(y,t)\,dy$$

and

$$K_1(x,t) = K(x,t)$$

The series form of $u(x)$ is

$$u(x) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \cdots$$

and since $u(x)$ is a Volterra equation,

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt$$

so

$$\begin{aligned} &u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \cdots = \\ &\quad f(x) + \lambda \int_a^x K(x,t)u_0(t)\,dt + \lambda^2 \int_a^x K(x,t)u_1(t)\,dt + \cdots \end{aligned}$$

Now, by equating like coefficients:

$$\begin{aligned} u_0(x) &= f(x) \\ u_1(x) &= \int_a^x K(x,t)u_0(t)\,dt \\ u_2(x) &= \int_a^x K(x,t)u_1(t)\,dt \\ \vdots \quad & \quad \vdots \\ u_n(x) &= \int_a^x K(x,t)u_{n-1}(t)\,dt \end{aligned}$$

Substituting $u_0(x) = f(x)$:

$$u_1(x) = \int_a^x K(x,t)f(t)\,dt$$

Substituting this for $u_1(x)$:

$$\begin{aligned} u_2(x) &= \int_a^x K(x,t)\int_a^s K(s,t)f(t)\,dt\,ds \\ &= \int_a^x f(t)\left[\int_t^x K(x,s)K(s,t)\,ds\right]dt \\ &= \int_a^x f(t)K^2(x,t)\,dt \end{aligned}$$

Continue substituting recursively for $u_3, u_4, \ldots$ to determine the resolvent kernel. Sometimes the resolvent kernel becomes recognizable after a few iterations as a Maclaurin or Taylor series and can be replaced by the function represented by that series. Otherwise numerical methods must be used to solve the equation.

Example: Use the Neumann series method to solve the Volterra integral equation of the 2nd kind:

$$u(x) = f(x) + \lambda\int_0^x e^{x-t}u(t)\,dt \tag{11}$$

In this example, $K_1(x,t) \equiv K(x,t) = e^{x-t}$, so $K_2(x,t)$ is found by:

$$\begin{aligned} K_2(x,t) &= \int_t^x K(x,s)K_1(s,t)\,ds \\ &= \int_t^x e^{x-s}e^{s-t}\,ds \\ &= \int_t^x e^{x-t}\,ds \\ &= (x-t)e^{x-t} \end{aligned}$$

and $K_3(x,t)$ is found by:

$$\begin{aligned} K_3(x,t) &= \int_t^x K(x,s)K_2(s,t)\,ds \\ &= \int_t^x e^{x-s}(s-t)e^{s-t}\,ds \end{aligned}$$

$$
\begin{aligned}
&= \int_t^x (s-t)e^{x-t}\,ds \\
&= e^{x-t}\int_t^x (s-t)\,ds \\
&= e^{x-t}\left[\frac{s^2}{2} - ts\right]_t^x \\
&= e^{x-t}\left[\frac{x^2-t^2}{2} - t(x-t)\right] \\
&= \frac{(x-t)^2}{2}e^{x-t}
\end{aligned}
$$

so by repeating this process the general equation for $K_{n+1}(x,t)$ is

$$K_{n+1}(x,t) = \frac{(x-t)^n}{n!}e^{x-t}.$$

Therefore, the resolvent kernel for eq. (11) is

$$
\begin{aligned}
\Gamma(x,t;\lambda) &= K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \cdots \\
&= e^{x-t} + \lambda(x-t)e^{x-t} + \lambda^2\frac{(x-t)^2}{2}e^{x-t} + \cdots \\
&= e^{x-t}\left[1 + \lambda(x-t) + \lambda^2\frac{(x-t)^2}{2} + \cdots\right].
\end{aligned}
$$

The series in brackets, $1 + \lambda(x-t) + \lambda^2\frac{(x-t)^2}{2} + \cdots$, is the Maclaurin series of $e^{\lambda(x-t)}$, so the resolvent kernel is

$$\Gamma(x,t;\lambda) = e^{x-t}e^{\lambda(x-t)} = e^{(\lambda+1)(x-t)}$$

and the solution to eq. (11) is

$$u(x) = f(x) + \lambda\int_0^x e^{(\lambda+1)(x-t)}f(t)\,dt.$$

5.2.2 Successive Approximations

1. make a reasonable 0th approximation, $u_0(x)$
2. let $i = 1$
3. $u_i = f(x) + \int_0^x K(x,t)u_{i-1}(t)\,dt$

4. increment i and repeat the previous step until a desired level of accuracy is reached

If $f(x)$ is continuous on $0 \leq x \leq a$ and $K(x,t)$ is continuous on both $0 \leq x \leq a$ and $0 \leq t \leq x$, then the sequence $u_n(x)$ will converge to $u(x)$. The sequence may occasionally be recognized as a Maclaurin or Taylor series, otherwise numerical methods must be used to solve the equation.

Example: Use the successive approximations method to solve the Volterra integral equation of the 2nd kind:

$$u(x) = x - \int_0^x (x-t)u(t)\,dt \tag{12}$$

In this example, start with the 0th approximation $u_0(t) = 0$. this gives $u_1(x) = x$,

$$\begin{aligned} u_2(x) &= x - \int_0^x (x-t)u_1(t)\,dt \\ &= x - \int_0^x (x-t)t\,dt \\ &= x - \left[\frac{xt^2}{2} - \frac{t^3}{3}\right]_0^x \\ &= x - \frac{x^3}{3!} \end{aligned}$$

and

$$\begin{aligned} u_3(x) &= x - \int_0^x (x-t)u_2(t)\,dt \\ &= x - \int_0^x (x-t)\left(t - \frac{t^3}{6}\right)\,dt \\ &= x - x\left[\frac{t^2}{2} - \frac{t^4}{24}\right]_0^x + \left[\frac{t^3}{3} - \frac{t^5}{30}\right]_0^x \\ &= x - x\left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \left(\frac{x^3}{3} - \frac{x^5}{30}\right) \\ &= x - \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^3}{3} - \frac{x^5}{30} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!}. \end{aligned}$$

Continuing this process, the nth approximation is

$$u_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

which is clearly the Maclaurin series of $\sin x$. Therefore the solution to eq. (12) is

$$u(x) = \sin x.$$

5.2.3 Laplace Transform

If the kernel depends only on $x-t$, then it is called a difference kernel and denoted by $K(x-t)$. Now the equation becomes:

$$u(x) = f(x) + \lambda \int_0^x K(x-t)u(t)\,dt,$$

which is the convolution product

$$\int_0^x K(x-t)u(t)\,dt = K * u$$

Define the Laplace transform pairs $U(s) \sim u(x)$, $F(s) \sim f(x)$, and $\mathcal{K}(s) \sim K(x)$. Now, since

$$\mathcal{L}\{K * u\} = \mathcal{K}U,$$

then performing the Laplace transform to the integral equations results in

$$\begin{aligned} U(s) &= F(s) + \lambda\mathcal{K}(s)U(s) \\ U(s) &= \frac{F(s)}{1-\lambda\mathcal{K}(s)}, \lambda\mathcal{K}(s) \neq 1, \end{aligned}$$

and performing the inverse Laplace transform:

$$u(x) = \mathcal{L}^{-1}\left\{\frac{F(S)}{1-\lambda\mathcal{K}(s)}\right\}, \lambda\mathcal{K}(s) \neq 1.$$

Example: Use the Laplace transform method to solve eq. (11):

$$u(x) = f(x) + \lambda \int_0^x e^{x-t}u(t)\,dt.$$

Let $\mathcal{K}(s) = \mathcal{L}\{e^x\} = \frac{1}{s-1}$ and

$$\begin{aligned} U(s) &= \frac{F(s)}{1-\frac{\lambda}{s-1}} \\ &= F(s)\frac{s-1}{s-1-\lambda} \end{aligned}$$

$$
\begin{aligned}
&= F(s)\frac{s-1-\lambda+\lambda}{s-1-\lambda} \\
&= F(s)+\lambda\frac{F(s)}{s-1-\lambda} \\
&= F(s)+\lambda\frac{F(s)}{s-(\lambda+1)}.
\end{aligned}
$$

The solution $u(x)$ is the inverse Laplace transform of $U(s)$:

$$
\begin{aligned}
u(x) &= \mathcal{L}^{-1}\left\{F(s)+\lambda\frac{F(s)}{s-(\lambda+1)}\right\} \\
&= \mathcal{L}^{-1}\{F(s)\}+\lambda\mathcal{L}^{-1}\left\{\frac{F(s)}{s-(\lambda+1)}\right\} \\
&= f(x)+\lambda\mathcal{L}^{-1}\left\{\frac{1}{s-(\lambda+1)}F(s)\right\} \\
&= f(x)+\lambda e^{(\lambda+1)x} * f(x) \\
&= f(x)+\lambda\int_0^x e^{(\lambda+1)(x-t)}f(t)\,dt
\end{aligned}
$$

which is the same result obtained using the Neumann series method.

Example: Use the Laplace transform method to solve eq. (12):

$$
u(x) = x - \int_0^x (x-t)u(t)\,dt.
$$

Let $\mathcal{K}(s) = \mathcal{L}\{x\} = \frac{1}{s^2}$ and

$$
\begin{aligned}
U(s) &= \frac{1}{s^2} - \frac{1}{s^2}U(s) \\
&= \frac{\frac{1}{s^2}}{1+\frac{1}{s^2}} \\
&= \frac{1}{s^2+1}
\end{aligned}
$$

The solution $u(x)$ is the inverse Laplace transform of $U(s)$:

$$
\begin{aligned}
u(x) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= \sin x
\end{aligned}
$$

which is the same result obtained using the successive approximations method.

5.2.4 Numerical Solutions

Given a 2nd kind Volterra equation

$$u(x) = f(x) + \int_a^x K(x,t)u(t)\,dt$$

divide the interval of integration (a, x) into n equal subintervals, $\Delta t = \frac{x_n - a}{n}.n \geq 1$, where $x_n = x$.

Let $t_0 = a$, $x_0 = t_0 = a$, $x_n = t_n = x$, $t_j = a + j\Delta t = t_0 + j\Delta t$, $x_i = x_0 + i\Delta t = a + i\Delta t = t_i$. Using the trapezoid rule,

$$\begin{aligned}\int_a^x K(x,t)u(t)\,dt \approx \\ \Delta t \Bigg[\frac{1}{2}K(x,t_0)u(t_0) + K(x,t_1)u(t_1) + \cdots \\ + K(x,t_{n-1})u(t_{n-1}) + \frac{1}{2}K(x,t_n)u(t_n)\Bigg]\end{aligned}$$

where $\Delta t = \frac{t_j - a}{j} = \frac{x-a}{n}$, $t_j \leq x$, $j \geq 1$, $x = x_n = t_n$.

Now the equation becomes

$$\begin{aligned}u(x) = f(x) + \Delta t \Bigg[\frac{1}{2}K(x,t_0)u(t_0) + K(x,t_1)u(t_1) + \cdots \\ + K(x,t_{n-1})u(t_{n-1}) + \frac{1}{2}K(x,t_n)u(t_n)\Bigg],\ t_j \leq x,\ j \geq 1,\ x = x_n = t_n.\end{aligned}$$

$\because K(x,t) \equiv 0$ when $t > x$ (the integration ends at $t = x$), $\therefore K(x_i, t_j) = 0$ for $t_j > x_i$. Numerically, the equation becomes

$$\begin{aligned}u(x_i) = f(x_i) + \Delta t \Bigg[\frac{1}{2}K(x_i,t_0)u(t_0) + K(x_i,t_1)u(t_1) + \cdots \\ + K(x_i,t_{j-1})u(t_{j-1}) + \frac{1}{2}K(x_i,t_j)u(t_j)\Bigg],\ i = 1,2,\ldots,n,\ t_j \leq x_i\end{aligned}$$

where $u(x_0) = f(x_0)$.

Denote $u_i \equiv u(t_i)$, $f_i \equiv f(x_i)$, $K_{ij} \equiv K(x_i, t_j)$, so the numeric equation can be written in a condensed form as

$$u_0 = f_0, u_i = f_i + \Delta t \left[\frac{1}{2} K_{i0} u_0 + K_{i1} u_1 + \cdots \right.$$
$$\left. + K_{i(j-1)} u_{j-1} + \frac{1}{2} K_{ij} u_j \right], \; i = 1, 2, \ldots, n, \; j \leq i$$

$\therefore$ there are $n+1$ equations

$$\begin{aligned}
u_0 &= f_0 \\
u_1 &= f_1 + \Delta t \left[\frac{1}{2} K_{10} u_0 + \frac{1}{2} K_{11} u_1 \right] \\
u_2 &= f_2 + \Delta t \left[\frac{1}{2} K_{20} u_0 + K_{21} u_1 + \frac{1}{2} K_{22} u_2 \right] \\
\vdots \quad & \quad \vdots \\
u_n &= f_n + \Delta t \left[\frac{1}{2} K_{n0} u_0 + K_{n1} u_1 + \cdots + K_{n(n-1)} u_{n-1} + \frac{1}{2} K_{nn} u_n \right]
\end{aligned}$$

where a general relation can be rewritten as

$$u_i = \frac{f_i + \Delta t \left[\frac{1}{2} K_{i0} u_0 + K_{i1} u_1 + \cdots + K_{i(i-1)} u_{i-1} \right]}{1 - \frac{\Delta t}{2} K_{ii}}$$

and can be evaluated by substituting $u_0, u_1, \ldots, u_{i-1}$ recursively from previous calculations.

5.3 Solving 1st-Kind Volterra Equations

5.3.1 Reduction to 2nd-Kind

$$f(x) = \lambda \int_0^x K(x,t) u(t) \, dt$$

If $K(x,x) \neq 0$, then:

$$\begin{aligned}
\frac{df}{dx} &= \lambda \int_0^x \frac{\partial K(x,t)}{\partial x} \mu(t) \, dt + \lambda K(x,x) u(x) \\
\Rightarrow \quad u(x) &= \frac{1}{\lambda K(x,x)} \frac{df}{dx} - \int_0^x \frac{1}{K(x,x)} \frac{\partial K(x,t)}{\partial x} u(t) \, dt
\end{aligned}$$

$$\text{Let} \quad g(x) = \frac{1}{\lambda K(x,x)} \text{ and } H(x,t) = \frac{-1}{K(x,x)} \frac{\partial K(x,t)}{\partial x}.$$
$$\therefore \quad u(x) = g(x) + \int_0^x H(x,t)u(t)\,dt$$

This is a Volterra equation of the 2nd kind.

Example: Reduce the following Volterra equation of the 1st kind to an equation of the 2nd kind and solve it:

$$\sin x = \int_0^x e^{x-t} u(t)\,dt. \tag{13}$$

In this case, $\lambda = 1$, $K(x,t) = e^{x-t}$, and $f(x) = \sin x$. Since $K(x,x) = 1 \neq 0$, then eq. (13) becomes:

$$u(x) = \cos x - \int_0^x e^{x-t} u(t)\,dt.$$

This is a special case of eq. (11) with $f(x) = \cos x$ and $\lambda = -1$:

$$\begin{aligned} u(x) &= \cos x - \int_0^x \cos t\,dt \\ &= \cos x - [\sin t]_0^x \\ &= \cos x - \sin x. \end{aligned}$$

5.3.2 Laplace Transform

If $K(x,t) = K(x-t)$, then a Laplace transform may be used to solve the equation. Define the Laplace transform pairs $U(s) \sim u(x)$, $F(s) \sim f(x)$, and $\mathcal{K}(s) \sim K(x)$.

Now, since

$$\mathcal{L}\{K * u\} = \mathcal{K}U,$$

then performing the Laplace transform to the integral equations results in

$$\begin{aligned} F(s) &= \lambda \mathcal{K}(s) U(s) \\ U(s) &= \frac{F(s)}{\lambda \mathcal{K}(s)}, \lambda \mathcal{K}(s) \neq 0, \end{aligned}$$

and performing the inverse Laplace transform:

$$u(x) = \frac{1}{\lambda} \mathcal{L}^{-1} \left\{ \frac{F(S)}{\mathcal{K}(s)} \right\}, \lambda \mathcal{K}(s) \neq 0.$$

Example: Use the Laplace transform method to solve eq. (13):

$$\sin x = \lambda \int_0^x e^{x-t} u(t)\, dt.$$

Let $\mathcal{K}(s) = \mathcal{L}\{e^x\} = \frac{1}{s-1}$ and substitute this and $\mathcal{L}\{\sin x\} = \frac{1}{s^2+1}$ to get

$$\frac{1}{s^2+1} = \lambda \frac{1}{s-1} U(s).$$

Solve for $U(s)$:

$$\begin{aligned} U(s) &= \frac{1}{\lambda} \frac{\frac{1}{s^2+1}}{\frac{1}{s-1}} \\ &= \frac{1}{\lambda} \frac{s-1}{s^2+1} \\ &= \frac{1}{\lambda} \left(\frac{s}{s^2+1} - \frac{1}{s^2+1} \right). \end{aligned}$$

Use the inverse transform to find $u(x)$:

$$\begin{aligned} u(x) &= \frac{1}{\lambda} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+1} - \frac{1}{s^2+1} \right\} \\ &= \frac{1}{\lambda} (\cos x - \sin x) \end{aligned}$$

which is the same result found using the reduction to 2nd kind method when $\lambda = 1$.

Example: Abel's equation (2) formulated as an integral equation is

$$-\sqrt{2g} f(x) = \int_0^x \frac{\phi(t)}{\sqrt{x-t}}\, dt.$$

This describes the shape $\phi(y)$ of a wire in a vertical plane along which a bead must descend under the influence of gravity a distance y_0 in a predetermined time $f(y_0)$. This problem is described by Abel's equation (2).

To solve Abel's equation, let $F(s)$ be the Laplace transform of $f(x)$ and $\Phi(s)$ be the Laplace transform of $\phi(x)$. Also let $\mathcal{K}(s) = \mathcal{L}\{\frac{1}{\sqrt{x}}\} = \sqrt{\frac{\pi}{s}}$ ($\because \Gamma(\frac{1}{2}) = \sqrt{\pi}$). The Laplace transform of eq. (2) is

$$-\sqrt{2g} F(s) = \sqrt{\frac{\pi}{s}} \Phi(s)$$

or, solving for $\Phi(s)$:

$$\Phi(s) = -\sqrt{\frac{2g}{\pi}}\sqrt{s}F(s).$$

So $\phi(x)$ is given by:

$$\phi(x) = -\sqrt{\frac{2g}{\pi}}\mathcal{L}^{-1}\{\sqrt{s}F(s)\}.$$

Since $\mathcal{L}^{-1}\{\sqrt{s}\}$ does not exist, the convolution theorem cannot be used directly to solve for $\phi(x)$. However, by introducing $H(s) \equiv \frac{F(s)}{\sqrt{s}}$ the equation may be rewritten as:

$$\phi(x) = -\sqrt{\frac{2g}{\pi}}sH(s).$$

Now

$$\begin{aligned} h(x) &= \mathcal{L}^{-1}\{H(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}F(s)\right\} \\ &= \frac{1}{\sqrt{\pi}}\int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt \end{aligned}$$

and since $h(0) = 0$,

$$\begin{aligned} \frac{dh}{dx} &= \mathcal{L}^{-1}\{sH(s) - h(0)\} \\ &= \mathcal{L}^{-1}\{sH(s)\} \end{aligned}$$

Finally, use $\frac{dh}{dx}$ to find $\phi(x)$ using the inverse Laplace transform:

$$\begin{aligned} \phi(x) &= -\sqrt{\frac{2g}{\pi}}\frac{d}{dx}\left[\frac{1}{\sqrt{\pi}}\int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt\right] \\ &= \frac{-\sqrt{2g}}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt \end{aligned}$$

which is the solution to Abel's equation (2).

6 Fredholm Equations

Fredholm integral equations are a type of linear integral equation (1) with $b(x) = b$:

$$h(x)u(x) = f(x) + \int_a^b K(x,t)u(t)\,dt \tag{14}$$

The equation is said to be the first kind when $h(x) = 0$,

$$-f(x) = \int_a^x K(x,t)u(t)\,dt \tag{15}$$

and the second kind when $h(x) = 1$,

$$u(x) = f(x) + \int_a^x K(x,t)u(t)\,dt \tag{16}$$

6.1 Relation to Boundary Value Problems

A general boundary value problem is given by:

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = g(x) \tag{17}$$

$$y(a) = c_1,\ y(b) = c_2$$

$$\therefore y(x) = f(x) + \int_a^b K(x,t)y(t)\,dt$$

where

$$f(x) = c_1 + \int_a^x (x-t)g(t)\,dt + \frac{x-a}{b-a}\left[c_2 - c_1 - \int_a^b (b-t)g(t)\,dt\right]$$

and

$$K(x,t) = \begin{cases} \frac{x-b}{b-a}\left[A(t) - (a-t)\left[A'(t) - B(t)\right]\right] & \text{if } x > t \\ \frac{x-a}{b-a}\left[A(t) - (b-t)\left[A'(t) - B(t)\right]\right] & \text{if } x < t \end{cases}$$

6.2 Fredholm Equations of the 2nd Kind With Seperable Kernels

A seperable kernel, also called a degenerate kernel, is a kernel of the form

$$K(x,t) = \sum_{k=1}^{n} a_k(x)b_k(t) \tag{18}$$

which is a finite sum of products $a_k(x)$ and $b_k(t)$ where $a_k(x)$ is a function of x only and $b_k(t)$ is a function of t only.

6.2.1 Nonhomogeneous Equations

Given the kernel in eq. (18), the general nonhomogeneous second kind Fredholm equation,

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt \tag{19}$$

becomes

$$\begin{aligned} u(x) &= f(x) + \lambda \int_a^b \sum_{k=1}^n a_k(x)b_k(t)u(t)\,dt \\ &= f(x) + \lambda \sum_{k=1}^n a_k(x) \int_a^b b_k(t)u(t)\,dt \end{aligned}$$

Define c_k as

$$c_k = \int_a^b b_k(t)u(t)\,dt$$

(i.e. the integral portion). Multiply both sides by $b_m(x)$, $m = 1, 2, \ldots, n$

$$b_m(x)u(x) = b_m(x)f(x) + \lambda \sum_{k=1}^n c_k b_m(x)a_k(x)$$

then integrate both sides from a to b

$$\int_a^b b_m(x)u(x)\,dx = \int_a^b b_m(x)f(x)\,dx + \lambda \sum_{k=1}^n c_k \int_a^b b_m(x)a_k(x)\,dx$$

Define f_m and a_{mk} as

$$f_m = \int_a^b b_m(x)f(x)\,dx$$

$$a_{mk} = \int_a^b b_m(x)a_k(x)\,dx$$

so finally the equation becomes

$$c_m = f_m + \lambda \sum_{k=1}^n a_{mk}c_k,\ m = 1, 2, \ldots, n.$$

That is, it is a nonhomogeneous system of n linear equations in $c_1, c_2, \ldots, c_n$. f_m and a_{mk} are known since $b_m(x)$, $f(x)$, and $a_k(x)$ are all given.

This system can be solved by finding all $c_m,\ m = 1, 2, \ldots, n$, and substituting into $u(x) = f(x) + \lambda \sum_{k=1}^{n} c_k a_k(x)$ to find $u(x)$. In matrix notation,

$$\text{let } C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},\ F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix},\ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

$$\therefore C = F + \lambda AC$$

$$C - \lambda AC = F$$

$$(I - \lambda A)C = F$$

which has a unique solution if the determinant $|I - \lambda A| \neq 0$, and either no solution or infinitely many solutions when $|I - \lambda A| = 0$.

Example: Use the above method to solve the Fredholm integral equation

$$u(x) = x + \lambda \int_0^1 (xt^2 + x^2 t) u(t)\, dt. \tag{20}$$

The kernel is $K(x,t) = xt^2 + x^2 t = \sum_{k=1}^{2} a_k(x) b_k(t)$, where $a_1(x) = x$, $a_2(x) = x^2$, $b_1(t) = t^2$, and $b_2(t) = t$. Given that $f(x) = x$, define f_1 and f_2:

$$\begin{aligned} f_1 &= \int_0^1 b_1(t) f(t)\, dt = \int_0^1 t^3\, dt = \frac{1}{4} \\ f_2 &= \int_0^1 b_2(t) f(t)\, dt = \int_0^1 t^2\, dt = \frac{1}{3} \end{aligned}$$

so that the matrix F is

$$F = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \end{bmatrix}.$$

Now find the elements of the matrix A:

$$\begin{aligned} a_{11} &= \int_0^1 b_1(t) a_1(t)\, dt = \int_0^1 t^3\, dt = \frac{1}{4} \\ a_{12} &= \int_0^1 b_1(t) a_2(t)\, dt = \int_0^1 t^4\, dt = \frac{1}{5} \\ a_{21} &= \int_0^1 b_2(t) a_1(t)\, dt = \int_0^1 t^2\, dt = \frac{1}{3} \\ a_{22} &= \int_0^1 b_2(t) a_2(t)\, dt = \int_0^1 t^3\, dt = \frac{1}{4} \end{aligned}$$

so that the A is

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}.$$

Therefore, $C = F + \lambda AC$ becomes

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \end{bmatrix} + \lambda \begin{bmatrix} \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or:

$$\begin{bmatrix} 1 - \frac{\lambda}{4} & -\frac{\lambda}{5} \\ -\frac{\lambda}{3} & 1 - \frac{\lambda}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \end{bmatrix}.$$

If the determinant of the leftmost term is not equal to zero, then there is a unique solution for c_1 and c_2 that can be evaluated by finding the inverse of that term. That is, if

$$\begin{aligned} \begin{vmatrix} 1 - \frac{\lambda}{4} & -\frac{\lambda}{5} \\ -\frac{\lambda}{3} & 1 - \frac{\lambda}{4} \end{vmatrix} &\neq 0 \\ \left(1 - \frac{\lambda}{4}\right)^2 - \frac{\lambda^2}{15} &\neq 0 \\ \frac{240 - 120\lambda - \lambda^2}{240} &\neq 0 \\ 240 - 120\lambda - \lambda^2 &\neq 0. \end{aligned}$$

If this is true, then c_1 and c_2 can be found as:

$$\begin{aligned} c_1 &= \frac{60 + \lambda}{240 - 120\lambda - \lambda^2} \\ c_2 &= \frac{80}{240 - 120\lambda - \lambda^2}, \end{aligned}$$

then $u(x)$ is found from

$$\begin{aligned} u(x) &= f(x) + \lambda \sum_{k=1}^{n} c_k a_k(x) \\ &= x + \lambda \sum_{k=1}^{2} c_k a_k(x) \\ &= x + \lambda \left[c_1 a_1(x) + c_2 a_2(x) \right] \\ &= x + \lambda \left[\frac{(60 + \lambda)x}{240 - 120\lambda - \lambda^2} + \frac{80x^2}{240 - 120\lambda - \lambda^2} \right] \end{aligned}$$

$$= \frac{(240-60\lambda)x+80\lambda x^2}{240-120\lambda-\lambda^2}, 240-120\lambda-\lambda^2 \neq 0.$$

6.2.2 Homogeneous Equations

Given the kernel defined in eq. (18), the general homogeneous second kind Fredholm equation,

$$u(x) = \lambda \int_a^b K(x,t)u(t)\,dt \tag{21}$$

becomes

$$\begin{aligned} u(x) &= \lambda \int_a^b \sum_{k=1}^n a_k(x)b_k(t)u(t)\,dt \\ &= \lambda \sum_{k=1}^n a_k(x) \int_a^b b_k(t)u(t)\,dt \end{aligned}$$

Define c_k, multiply and integrate, then define a_{mk} as in the previous section so finally the equation becomes

$$c_m = \lambda \sum_{k=1}^n a_{mk}c_k,\ m = 1,2,\ldots,n.$$

That is, it is a homogeneous system of n linear equations in $c_1, c_2, \ldots, c_n$. Again, the system can be solved by finding all $c_m,\ m = 1,2,\ldots,n$, and substituting into $u(x) = \lambda \sum_{k=1}^n c_k a_k(x)$ to find $u(x)$. Using the same matrices defined earlier, the equation can be written as

$$C = \lambda AC$$

$$C - \lambda AC = 0$$

$$(I - \lambda A)C = 0.$$

Because this is a homogeneous system of equations, when the determinant $|I - \lambda A| \neq 0$ then the trivial solution $C = 0$ is the only solution, and the solution to the integral equation is $u(x) = 0$. Otherwise, if $|I - \lambda A| = 0$ then there may be zero or infinitely many solutions whose eigenfunctions correspond to solutions $u_1(x), u_2(x), \ldots, u_n(x)$ of the Fredholm equation.

Example: Use the above method to solve the Fredholm integral equation

$$u(x) = \lambda \int_0^\pi (\cos^2 x \cos 2t + \cos 3x \cos^3 t)u(t)\,dt. \tag{22}$$

The kernel is $K(x,t) = \cos^2 x \cos 2t + \cos 3x \cos^3 t = \sum_{k=1} 2a_k(x)b_k(t)$ where $a_1(x) = \cos^2 x$, $a_2(x) = \cos 3x$, $b_1(t) = \cos 2t$, and $b_2(t) = \cos^3 t$. Now find the elements of the matrix A:

$$\begin{aligned}
a_{11} &= \int_0^\pi b_1(t)a_1(t)\,dt = \int_0^\pi \cos 2t \cos^2 t\,dt = \frac{\pi}{4} \\
a_{12} &= \int_0^\pi b_1(t)a_2(t)\,dt = \int_0^\pi \cos 2t \cos 3t\,dt = 0 \\
a_{21} &= \int_0^\pi b_2(t)a_1(t)\,dt = \int_0^\pi \cos^3 t \cos^2 t\,dt = 0 \\
a_{22} &= \int_0^\pi b_2(t)a_2(t)\,dt = \int_0^\pi \cos^3 t \cos 3t\,dt = \frac{\pi}{8}
\end{aligned}$$

and follow the method of the previous example to find c_1 and c_2:

$$\begin{aligned}
\begin{bmatrix} 1-\frac{\lambda\pi}{4} & 0 \\ 0 & 1-\frac{\lambda\pi}{8} \end{bmatrix} &= 0 \\
\left(1-\lambda\frac{\pi}{4}\right) &= 0 \\
\left(1-\lambda\frac{\pi}{8}\right) &= 0
\end{aligned}$$

For c_1 and c_2 to not be the trivial solution, the determinant must be zero. That is,

$$\begin{vmatrix} 1-\frac{\lambda\pi}{4} & 0 \\ 0 & 1-\frac{\lambda\pi}{8} \end{vmatrix} = \left(1-\lambda\frac{\pi}{4}\right)\left(1-\lambda\frac{\pi}{8}\right) = 0,$$

which has solutions $\lambda_1 = \frac{4}{\pi}$ and $\lambda_2 = \frac{8}{\pi}$ which are the eigenvalues of eq. (22). There are two corresponding eigenfunctions $u_1(x)$ and $u_2(x)$ that are solutions to eq. (22). For $\lambda_1 = \frac{4}{\pi}$, $c_1 = c_1$ (i.e. it is a parameter), and $c_2 = 0$. Therefore $u_1(x)$ is

$$u_1(x) = \frac{4}{\pi}c_1 \cos^2 x.$$

To form a specific solution (which can be used to find a general solution, since eq. (22) is linear), let $c_1 = \frac{\pi}{4}$, so that

$$u_1(x) = \cos^2 x.$$

Similarly, for $\lambda_2 = \frac{8}{\pi}$, $c_1 = 0$ and $c_2 = c_2$, so $u_2(x)$ is

$$u_2(x) = \frac{8}{\pi}c_2 \cos 3x.$$

To form a specific solution, let $c_2 = \frac{\pi}{8}$, so that

$$u_2(x) = \cos 3x.$$

6.2.3 Fredholm Alternative

The homogeneous Fredholm equation (21),

$$u(x) = \lambda \int_a^b K(x,t)u(t)\,dt$$

has the corresponding nonhomogeneous equation (19),

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt.$$

If the homogeneous equation has only the trivial solution $u(x) = 0$, then the nonhomogeneous equation has exactly one solution. Otherwise, if the homogeneous equation has nontrivial solutions, then the nonhomogeneous equation has either no solutions or infinitely many solutions depending on the term $f(x)$. If $f(x)$ is orthogonal to every solution $u_i(x)$ of the homogeneous equation, then the associated nonhomogeneous equation will have a solution. The two functions $f(x)$ and $u_i(x)$ are orthogonal if $\int_a^b f(x)u_i(x)\,dx = 0$.

6.2.4 Approximate Kernels

Often a nonseperable kernel may have a Taylor or other series expansion that *is* separable. This approximate kernel is denoted by $M(x,t) \approx K(x,t)$ and the corresponding approximate solution for $u(x)$ is $v(x)$. So the approximate solution to eq. (19) is given by

$$v(x) = f(x) + \lambda \int_a^b M(x,t)v(t)\,dt.$$

The error involved is $\varepsilon = |u(x) - v(x)|$ and can be estimated in some cases.

Example: Use an approximating kernel to find an approximate solution to

$$u(x) = \sin x + \int_0^1 (1 - x\cos xt)u(t)\,dt. \tag{23}$$

The kernel $K(x,t) = 1 - x\cos xt$ is not seperable, but a finite number of terms from its Maclaurin series expansion

$$1 - x\left(1 - \frac{x^2t^2}{2!} + \frac{x^4t^4}{4!} - \cdots\right) = 1 - x + \frac{x^3t^2}{2!} - \frac{x^5t^4}{4!} + \cdots$$

is seperable in x and t. Therefore consider the first three terms of this series as the approximate kernel

$$M(x,t) = 1 - x + \frac{x^3t^2}{2!}.$$

The associated approximate integral equation in $v(x)$ is

$$v(x) = \sin x + \int_0^1 \left(1 - x + \frac{x^3 t^2}{2!}\right) v(t)\, dt. \tag{24}$$

Now use the previous method for solving Fredholm equations with seperable kernels to solve eq. (24):

$$M(x,t) = \sum_{k=1}^{2} a_k(x) b_k(t)$$

where $a_1(x) = (1-x)$, $a_2(x) = x^3$, $b_1(t) = 1$, $b_2(t) = \frac{t^2}{2}$. Given that $f(x) = \sin x$, find f_1 and f_2:

$$\begin{aligned} f_1 &= \int_0^1 b_1(t) f(t)\, dt = \int_0^1 \sin t\, dt = 1 - \cos 1 \\ f_2 &= \int_0^1 b_2(t) f(t)\, dt = \int_0^1 \frac{t^2}{2} \sin t\, dt = \frac{1}{2}\cos 1 + \sin 1 - 1. \end{aligned}$$

Now find the elements of the matrix A:

$$\begin{aligned} a_{11} &= \int_0^1 b_1(t) a_1(t)\, dt = \int_0^1 (1-t)\, dt = \frac{1}{2} \\ a_{12} &= \int_0^1 b_1(t) a_2(t)\, dt = \int_0^1 t^3\, dt = \frac{1}{4} \\ a_{21} &= \int_0^1 b_2(t) a_1(t)\, dt = \int_0^1 \frac{t^2}{2}(1-t)\, dt = \frac{1}{24} \\ a_{22} &= \int_0^1 b_2(t) a_2(t)\, dt = \int_0^1 \frac{t^5}{2}\, dt = \frac{1}{12}. \end{aligned}$$

Therefore, $C = F + \lambda AC$ becomes

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 - \cos 1 \\ \frac{1}{2}\cos 1 + \sin 1 - 1 \end{bmatrix} + 1 \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{24} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or

$$\begin{bmatrix} 1 - \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{24} & 1 - \frac{1}{12} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 - \cos 1 \\ \frac{1}{2}\cos 1 + \sin 1 - 1 \end{bmatrix}.$$

The determinant is

$$\begin{vmatrix} 1-\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{24} & 1-\frac{1}{12} \end{vmatrix} = \frac{1}{2}\cdot\frac{11}{12} - \frac{1}{4}\cdot\frac{1}{24} = \frac{43}{96}$$

which is non-zero, so there is a unique solution for C:

$$\begin{aligned} c_1 &= -\frac{76}{43}\cos 1 + \frac{64}{43} + \frac{24}{43}\sin 1 \approx 1.0031 \\ c_2 &= \frac{20}{43}\cos 1 - \frac{44}{43} + \frac{48}{43}\sin 1 \approx 0.1674. \end{aligned}$$

Therefore, the solution to eq. (24) is

$$\begin{aligned} v(x) &= \sin x + c_1 a_1(x) + c_2 a_2(x) \\ &\approx \sin x + 1.0031(1-x) + 0.1674x^3 \end{aligned}$$

which is an approximate solution to eq. (23). The exact solution is known to be $u(x) = 1$, so comparing some values of $v(x)$:

Solution	0	0.25	0.5	0.75	1
u(x)	1	1	1	1	1
v(x)	1.0031	1.0023	1.0019	1.0030	1.0088

6.3 Fredholm Equations of the 2nd Kind With Symmetric Kernels

A symmetric kernel is a kernel that satisfies

$$K(x,t) = K(t,x).$$

If the kernel is a complex-valued function, then to be symmetric it must satisfy $[K(x,t) = \overline{K(t,x)}]$ where $\overline{K}$ denotes the complex conjugate of K.

These types of integral equations may be solved by using a resolvent kernel $\Gamma(x,t;\lambda)$ that can be found from the orthonormal eigenfunctions of the homogeneous form of the equation with a symmetric kernel.

6.3.1 Homogeneous Equations

The eigenfunctions of a homogeneous Fredholm equation with a symmetric kernel are functions $u_n(x)$ that satisfy

$$u_n(x) = \lambda_n \int_a^b K(x,t)u_n(t)\,dt.$$

It can be shown that the eigenvalues of the symmetric kernel are real, and that the eigenfunctions $u_n(x)$ and $u_m(x)$ corresponding to two of the eigenvalues λ_n and λ_m where $\lambda_n \neq \lambda_m$ are orthogonal (see earlier section on the Fredholm Alternative). It can also be shown that there is a finite number of eigenfunctions corresponding to each eigenvalue of a kernel if the kernel is square integrable on $\{(x,t)|a \leq x \leq b, a \leq t \leq b\}$, that is:

$$\int_a^b \int_a^b K^2(x,t)\,dx\,dt = B^2 < \infty.$$

Once the orthonormal eigenfunctions $\phi_1(x), \phi_2(x), \ldots$ of the homogeneous form of the equations are found, the resolvent kernel can be expressed as an infinite series:

$$\Gamma(x,t;\lambda) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k(t)}{\lambda - \lambda_k}, \lambda \neq \lambda_k$$

so the solution of the 2nd-order Fredholm equation is

$$u(x) = f(x) + \lambda \sum_{k=1}^{\infty} \frac{a_k \phi_k(x)}{\lambda_k - \lambda}$$

where

$$a_k = \int_a^b f(x)\phi_k(x)\,dx.$$

6.4 Fredholm Equations of the 2nd Kind With General Kernels

6.4.1 Fredholm Resolvent Kernel

Evaluate the Fredholm Resolvent Kernel $\Gamma(x,t;\lambda)$ in

$$u(x) = f(x) + \lambda \int_a^b \Gamma(x,t;\lambda) f(t)\,dt \tag{25}$$

$$\Gamma(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}$$

where $D(x,t;\lambda)$ is called the Fredholm minor and $D(\lambda)$ is called the Fredholm determinant. The minor is defined as

$$D(x,t;\lambda) = K(x,t) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} B_n(x,t)$$

where

$$B_n(x,t) = C_n K(x,t) - n \int_a^b K(x,s) B_{n-1}(s,t)\, ds,\ B_0(x,t) = K(x,t)$$

and

$$C_n = \int_a^b B_{n-1}(t,t)\, dt, n = 1, 2, \ldots,\ C_0 = 1$$

and the determinant is defined as

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} C_n,\ C_0 = 1.$$

$B_n(x,t)$ and C_n may each be expressed as a repeated integral where the integrand is a determinant:

$$B_n(x,t) = \int_a^b \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,t) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,t) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(t_n,t) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1\, dt_2\, \ldots\, dt_n$$

$$C_n = \int_a^b \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & \cdots & K(t_1,t_n) \\ K(t_2,t_1) & K(t_2,t_2) & \cdots & K(t_2,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(t_n,t_1) & K(t_n,t_2) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1\, dt_2\, \ldots\, dt_n$$

6.4.2 Iterated Kernels

Define the iterated kernel $K_i(x,t)$ as

$$K_i(x,y) \equiv \int_a^b K(x,t) K_{i-1}(t,y)\, dt,\ K_1(x,t) \equiv K(x,t)$$

so

$$\phi_i(x) \equiv \int_a^b K_i(x,y) f(y)\, dy$$

and

$$\begin{aligned} u_n(x) &= f(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots + \lambda^n\phi_n(x) \\ &= f(x) + \sum_{i=1}^{n} \lambda^i \phi_i(x) \end{aligned}$$

$u_n(x)$ converges to $u(x)$ when $|\lambda B| < 1$ and

$$B = \sqrt{\int_a^b \int_a^b K^2(x,t)\,dx\,dt}.$$

6.4.3 Neumann Series

The Neumann series

$$u(x) = f(x) + \sum_{i=1}^{\infty} \lambda^i \phi_i(x)$$

is convergent, and substituting for $\phi_i(x)$ as in the previous section it can be rewritten as

$$\begin{aligned} u(x) &= f(x) + \sum_{i=1}^{\infty} \lambda^i \int_a^b K_i(x,t) f(t)\,dt \\ &= f(x) + \int_a^b \left[\sum_{i=1}^{\infty} \lambda^i K_i(x,t) \right] f(t)\,dt \\ &= f(x) + \lambda \int_a^b \Gamma(x,t;\lambda) f(t)\,dt. \end{aligned}$$

Therefore the Neumann resolvent kernel for the general nonhomogeneous Fredholm integral equation of the second kind is

$$\Gamma(x,t;\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_i(x,t).$$

6.5 Approximate Solutions To Fredholm Equations of the 2nd Kind

The solution of the general nonhomogeneous Fredholm integral equation of the second kind eq. (19), $u(x)$, can be approximated by the partial sum

$$S_N(x) = \sum_{k=1}^{N} c_k \phi_k(x) \tag{26}$$

of N linearly independent functions $\phi_1, \phi_2, \ldots, \phi_N$ on the interval (a, b). The associated error $\varepsilon(x, c_1, c_2, \ldots, c_N)$ depends on x and the choice of the coefficients $c_1, c_2, \ldots, c_N$. Therefore, when substituting the approximate solution for $u(x)$, the equation becomes

$$S_N(x) = f(x) + \int_a^b K(x,t) S_N(t)\, dt + \varepsilon(x, c_1, c_2, \ldots, c_N). \tag{27}$$

Now N conditions must be found to give the N equations that will determine the coefficients $c_1, c_2, \ldots, c_N$.

6.5.1 Collocation Method

Assume that the error term ε disappears at the N points $x_1, x_2, \ldots, x_N$. This reduces the equation to N equations:

$$S_N(x_i)\ f(x_i) + \int_a^b K(x_i, t) S_N(t)\, dt,\ i = 1, 2, \ldots, N,$$

where the coefficients of S_N can be found by substituting the N linearly independent functions $\phi_1, \phi_2, \ldots, \phi_N$ and values $x_1, x_2, \ldots, x_N$ where the error vanishes, then performing the integration and solving for the coefficients.

Example: Use the Collocation method to solve the Fredholm integral equation of the second kind

$$u(x) = x + \int_{-1}^{1} xtu(t)\, dt. \tag{28}$$

Set $N = 3$ and choose the linearly independent functions $\phi_1(x) = 1$, $\phi_2(x) = x$, and $\phi_3(x) = x^2$. Therefore the approximate solution is

$$S_3(x) = \sum_{k=1}^{3} c_k \phi_k(x) = c_1 + c_2 x + c_3 x^2.$$

Substituting into eq. (27) obtains

$$\begin{aligned} S_3(x) &= x + \int_{-1}^{1} xt(c_1 + c_2 t + c_3 t^2)\, dt + \varepsilon(x, c_1, c_2, c_3) \\ &= x + x \int_{-1}^{1} (c_1 t + c_2 t^2 + c_3 t^3)\, dt + \varepsilon(x, c_1, c_2, c_3) \end{aligned}$$

and performing the integration results in

$$\begin{aligned}
\int_{-1}^{1} (c_1 t + c_2 t^2 + c_3 t^3)\, dt &= \left[\frac{c_1 t^2}{2} + \frac{c_2 t^3}{3} + \frac{c_3 t^4}{4}\right]_{-1}^{1} \\
&= \frac{c_1}{2} + \frac{c_2}{3} + \frac{c_3}{4} - \left(\frac{c_1}{2} - \frac{c_2}{3} + \frac{c_3}{4}\right) \\
&= \frac{c_1 - c_1}{2} + \frac{c_2 + c_2}{3} + \frac{c_3 - c_3}{4} \\
&= \frac{2}{3} c_2
\end{aligned}$$

so eq. (27) becomes

$$\begin{aligned}
c_1 + c_2 x + c_3 x^2 &= x + x\left(\frac{2}{3} c_2\right) + \varepsilon(x, c_1, c_2, c_3) \\
&= x\left(1 + \frac{2}{3} c_2\right) + \varepsilon(x, c_1, c_2, c_3).
\end{aligned}$$

Now, three equations are needed to find c_1, c_2, and c_3. Assert that the error term is zero at three points, arbitrarily chosen to be $x_1 = 1$, $x_2 = 0$, and $x_3 = -1$. This gives

$$\begin{aligned}
c_1 + c_2 + c_3 &= 1 + \frac{2}{3} c_2 \\
c_1 + \frac{1}{3} c_2 + c_3 &= 1,
\end{aligned}$$

$$\begin{aligned}
c_1 + 0 + 0 &= 0 \\
c_1 &= 0,
\end{aligned}$$

$$\begin{aligned}
c_1 - c_2 + c_3 &= -1 - \frac{2}{3} c_2 \\
c_1 - \frac{1}{3} c_2 + c_3 &= -1.
\end{aligned}$$

Solve for c_1, c_2, and c_3, giving $c_1 = 0$, $c_2 = 3$, and $c_3 = 0$. Therefore the approximate solution to eq. (28) is $S_3(x) = 3x$. In this case, the exact solution is known to be $u(x) = 3x$, so the approximate solution happened to be equal to the exact solution due to the selection of the linearly independent functions.

6.5.2 Galerkin Method

Assume that the error term ε is orthogonal to N given linearly independent functions $\psi_1, \psi_2, \ldots, \psi_N$ of x on the interval (a, b). The N conditions therefore become

$$\begin{aligned}
&\int_a^b \psi_j(x)\varepsilon(x, c_1, c_2, \ldots, c_N)\, dx \\
&= \int_a^b \psi_j(x)\left[S_N(x) - f(x) - \int_a^b K(x,t)S_N(t)\, dt\right] dx \\
&= 0,\ j = 1, 2, \ldots, N
\end{aligned}$$

which can be rewritten as either

$$\begin{aligned}
&\int_a^b \psi_j(x)\left[S_N(x) - \int_a^b K(x,t)S_N(t)\, dt\right] dx \\
&= \int_a^b \psi_j(x)f(x)\, dx,\ j = 1, 2, \ldots, N
\end{aligned}$$

or

$$\begin{aligned}
&\int_a^b \psi_j(x)\left\{\sum_{k=1}^N c_k\phi_k(x) - \int_a^b K(x,t)\left[\sum_{k=1}^N c_k\phi_k(t)\right] dt\right\} dx \\
&= \int_a^b \psi_j(x)f(x)\, dx,\ j = 1, 2, \ldots, N
\end{aligned}$$

after substituting for $S_N(x)$ as before. In general the first set of linearly independent functions $\phi_j(x)$ are different from the second set $\psi_j(x)$, but the same functions may be used for convenience.

Example: Use the Galerkin method to solve the Fredholm integral equation in eq. (28),

$$u(x) = x + \int_{-1}^1 xtu(t)\, dt$$

using the same linearly independent functions $\phi_1(x) = 1$, $\phi_2(x) = x$, and $\phi_3(x) = x^2$ giving the approximate solution

$$S_3(x) = c_1 + c_2x + c_3x^2.$$

Substituting into eq. (27) results in the error term being

$$\varepsilon(x, c_1, c_2, c_3) = c_1 + c_2 x + c_3 x^2 - x - \int_{-1}^{1} xt(c_1 + c_2 t + c_3 t^2)\, dt.$$

The error must be orthogonal to three linearly independent functions, chosen to be $\psi_1(x) = 1$, $\psi_2(x) = x$, and $\psi_3(x) = x^2$:

$$\begin{aligned}
\int_{-1}^{1} 1 \left[c_1 + c_2 x + c_3 x^2 - \int_{-1}^{1} xt(c_1 + c_2 t + c_3 t^2)\, dt \right] dx &= \int_{-1}^{1} x\, dx \\
\int_{-1}^{1} \left[c_1 + c_2 x + c_3 x^2 - x \int_{-1}^{1} (c_1 t + c_2 t^2 + c_3 t^3)\, dt \right] dx &= \left. \frac{x^2}{2} \right|_{-1}^{1} \\
\int_{-1}^{1} \left[c_1 + c_2 x + c_3 x^2 - x \left[\frac{c_1 t^2}{2} + \frac{c_2 t^3}{3} + \frac{c_3 t^4}{4} \right]_{-1}^{1} \right] dx &= \frac{1}{2} - \frac{1}{2} \\
\int_{-1}^{1} \left(c_1 - \frac{1}{3} c_2 x + c_3 x^2 \right) dx &= 0 \\
\left[c_1 x + \frac{c_2 x^2}{6} + \frac{c_3 x^3}{3} \right]_{-1}^{1} &= 0 \\
2c_1 + \frac{2}{3} c_3 &= 0,
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{1} x \left[c_1 + c_2 x + c_3 x^2 - \int_{-1}^{1} xt(c_1 + c_2 t + c_3 t^2)\, dt \right] dx &= \int_{-1}^{1} x^2\, dx \\
\int_{-1}^{1} x \left(c_1 - \frac{1}{3} c_2 x + c_3 x^2 \right) dx &= \left. \frac{x^3}{3} \right|_{-1}^{1} \\
\int_{-1}^{1} \left(c_1 x - \frac{1}{3} c_2 x^2 + c_3 x^3 \right) dx &= \frac{1}{3} + \frac{1}{3} \\
\left[\frac{c_1 x^2}{2} + \frac{c_2 x^3}{9} + \frac{c_3 x^4}{4} \right]_{-1}^{1} &= \frac{2}{3} \\
\frac{2}{9} c_2 &= \frac{2}{3} \\
c_2 &= 3,
\end{aligned}$$

$$\int_{-1}^{1} x^2 \left[c_1 + c_2 x + c_3 x^2 - \int_{-1}^{1} xt(c_1 + c_2 t + c_3 t^2)\, dt \right] dx = \int_{-1}^{1} x^3\, dx$$

$$
\begin{aligned}
\int_{-1}^{1} x^2 \left(c_1 - \frac{1}{3}c_2 x + c_3 x^2 \right) dx &= \left. \frac{x^4}{4} \right|_{-1}^{1} \\
\int_{-1}^{1} \left(c_1 x^2 - \frac{1}{3}c_2 x^3 + c_3 x^4 \right) dx &= \frac{1}{4} - \frac{1}{4} \\
\left[\frac{c_1 x^3}{3} + \frac{c_2 x^4}{12} + \frac{c_3 x^5}{5} \right]_{-1}^{1} &= 0 \\
\frac{2}{3}c_1 + \frac{2}{5}c_3 &= 0.
\end{aligned}
$$

Solving for c_1, c_2, and c_3 gives $c_1 = 0$, $c_2 = 3$, and $c_3 = 0$, resulting in the approximate solution being $S_3(x) = 3x$. This is the same result obtained using the collocation method, and also happens to be the exact solution to eq. (27) due to the selection of the linearly independent functions.

6.6 Numerical Solutions To Fredholm Equations

A numerical solution to a general Fredholm integral equation can be found by approximating the integral by a finite sum (using the trapezoid rule) and solving the resulting simultaneous equations.

6.6.1 Nonhomogeneous Equations of the 2nd Kind

In the general Fredholm Equation of the second kind in eq. (19),

$$u(x) = f(x) + \int_a^b K(x,t)u(t)\,dt,$$

the interval (a, b) can be subdivided into n equal intervals of width $\Delta t = \frac{b-a}{n}$. Let $t_0 = a$, $t_j = a + j\Delta t = t_0 + j\Delta t$, and since the variable is either t or x, let $x_0 = t_0 = a$, $x_n = t_n = b$, and $x_i = x_0 + i\Delta t$ (i.e. $x_i = t_i$). Also denote $u(x_i)$ as u_i, $f(x_i)$ as f_i, and $K(x_i, t_j)$ as K_{ij}. Now if the trapezoid rule is used to approximate the given equation, then:

$$
\begin{aligned}
u(x) = f(x) + \int_a^b K(x,t)u(t)\,dt \approx f(x) + \Delta t \Bigg[&\frac{1}{2}K(x,t_0)u(t_0) \\
&+ K(x,t_1)u(t_1) + \cdots + K(x,t_{n-1})u(t_{n-1}) + \frac{1}{2}K(x,t_n)u(t_n) \Bigg]
\end{aligned}
$$

or, more tersely:

$$u(x) \approx f(x) + \Delta t \left[\frac{1}{2} K(x, t_0) u_0 + K(x, t_1) u_1 + \cdots + \frac{1}{2} K(x, t_n) u_n \right].$$

There are $n+1$ values of u_i, as $i = 0, 1, 2, \ldots, n$. Therefore the equation becomes a set of $n+1$ equations in u_i:

$$\begin{aligned} u_i = f_i + \Delta t \Bigg[& \frac{1}{2} K_{i0} u_0 + K_{i1} u_1 + \cdots + K_{i(n-1)} u_{n-1} \\ & + \frac{1}{2} K_{in} u_n \Bigg], \ i = 0, 1, 2, \ldots, n \end{aligned}$$

that give the approximate solution to $u(x)$ at $x = x_i$. The terms involving u may be moved to the left side of the equations, resulting in $n+1$ equations in $u_0, u_1, u_2, \ldots, u_n$:

$$\begin{aligned} \left(1 - \frac{\Delta t}{2} K_{00}\right) u_0 - \Delta t K_{01} u_1 - \Delta t K_{02} u_2 - \cdots - \frac{\Delta t}{2} K_{0n} u_n &= f_0 \\ -\frac{\Delta t}{2} K_{10} u_0 + (1 - \Delta t K_{11}) u_1 - \Delta t K_{12} u_2 - \cdots - \frac{\Delta t}{2} K_{1n} u_n &= f_1 \\ \vdots \qquad & \vdots \\ -\frac{\Delta t}{2} K_{n0} u_0 - \Delta t K_{n1} u_1 - \Delta t K_{n2} u_2 - \cdots + \left(1 - \frac{\Delta t}{2} K_{nn}\right) u_n &= f_n. \end{aligned}$$

This may also be written in matrix form:

$$KU = F,$$

where K is the matrix of coefficients

$$K = \begin{bmatrix} 1 - \frac{\Delta t}{2} K_{00} & -\Delta t K_{01} & \cdots & -\frac{\Delta t}{2} K_{0n} \\ -\frac{\Delta t}{2} K_{10} & 1 - \Delta t K_{11} & \cdots & -\frac{\Delta t}{2} K_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\Delta t}{2} K_{n0} & -\Delta t K_{n1} & \cdots & 1 - \frac{\Delta t}{2} K_{nn} \end{bmatrix},$$

U is the matrix of solutions

$$U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix} c,$$

and F is the matrix of the nonhomogeneous part

$$F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

Clearly there is a unique solution to this system of linear equations when $|K| \neq 0$, and either infinite or zero solutions when $|K| = 0$.

Example: Use the trapezoid rule to find a numeric solution to the integral equation in eq. (23):

$$u(x) = \sin x + \int_0^1 (1 - x\cos xt)u(t)\,dt$$

at $x = 0$, $\frac{1}{2}$, and 1. Choose $n = 2$ and $\Delta t = \frac{1-0}{2} = \frac{1}{2}$, so $t_i = i\Delta t = \frac{i}{2} = x_i$. Using the trapezoid rule results in

$$u_i = f_i + \frac{1}{2}\left(\frac{1}{2}K_{i0}u_0 + K_{i1}u_1 + \frac{1}{2}K_{i2}u_2\right), i = 0, 1, 2,$$

or in matrix form:

$$\begin{bmatrix} 1 - \frac{1}{4}K_{00} & -\frac{1}{2}K_{01} & -\frac{1}{4}K_{02} \\ -\frac{1}{4}K_{10} & 1 - \frac{1}{2}K_{11} & -\frac{1}{4}K_{12} \\ -\frac{1}{4}K_{20} & -\frac{1}{2}K_{21} & 1 - \frac{1}{4}K_{22} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sin 0 \\ \sin\frac{1}{2} \\ \sin 1 \end{bmatrix}.$$

Substituting $f_i = f(x_i) = \sin\frac{i}{2}$ and $K_{ij} = K(x_i, t_j) = 1 - \frac{i}{2}\cos\frac{ij}{4}$ into the matrix obtains

$$\begin{bmatrix} 1 - \frac{1}{4}(1-0) & -\frac{1}{2}(1-0) & -\frac{1}{4}(1-0) \\ -\frac{1}{4}\left(1 - \frac{1}{2}\right) & 1 - \frac{1}{2}\left(1 - \frac{1}{2}\cos\frac{1}{4}\right) & -\frac{1}{4}\left(1 - \frac{1}{2}\cos\frac{1}{2}\right) \\ -\frac{1}{4}(1-1) & -\frac{1}{2}\left(1 - \cos\frac{1}{2}\right) & 1 - \frac{1}{4}(1 - \cos 1) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sin 0 \\ \sin\frac{1}{2} \\ \sin 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{8} & \frac{1}{2} + \frac{1}{4}\cos\frac{1}{4} & -\frac{1}{4} + \frac{1}{8}\cos\frac{1}{2} \\ 0 & -\frac{1}{2} + \frac{1}{2}\cos\frac{1}{2} & \frac{3}{4} + \frac{1}{4}\cos 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sin 0 \\ \sin\frac{1}{2} \\ \sin 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.75 & -0.5 & -0.25 \\ -0.125 & 0.7422 & -0.1403 \\ 0 & -0.0612 & 0.8851 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.4794 \\ 0.8415 \end{bmatrix},$$

which can be solved to give $u_0 \approx 1.0132$, $u_1 \approx 1.0095$, and $u_2 \approx 1.0205$. The exact solution of eq. (23) is known to be $u(x) = 1$, which compares very well with these approximate values.

6.6.2 Homogeneous Equations

The general homogeneous Fredholm equation in eq. (21),

$$u(x) = \lambda \int_a^b K(x,t)u(t)\,dt$$

is solved in a similar manner as nonhomogeneous Fredholm equations using the trapezoid rule. Using the same terminology as the previous section, the integral is divided into n equal subintervals of width Δt, resulting in $n+1$ linear homogeneous equations

$$u_i = \lambda \Delta t \left[\frac{1}{2} K_{i0} u_0 + K_{i1} u_1 + \cdots + K_{i(n-1)} u_{n-1} + \frac{1}{2} K_{in} u_n \right], \; i = 0, 1, \ldots, n.$$

Bringing all of the terms to the left side of the equations results in

$$\begin{aligned}
\left(1 - \frac{\lambda \Delta t}{2} K_{00}\right) u_0 - \lambda \Delta t K_{01} u_1 - \cdots - \frac{\lambda \Delta t}{2} K_{0n} u_n &= 0 \\
-\frac{\lambda \Delta t}{2} K_{10} u_0 + (1 - \lambda \Delta t K_{11})\, u_1 - \cdots - \frac{\lambda \Delta t}{2} K_{1n} u_n &= 0 \\
\vdots \qquad & \vdots \\
-\frac{\lambda \Delta t}{2} K_{n0} u_0 - \lambda \Delta t K_{n1} u_1 - \cdots + \left(1 - \frac{\lambda \Delta t}{2} K_{nn}\right) u_n &= 0.
\end{aligned}$$

Letting $\lambda = \frac{1}{\mu}$ simplifies this system by causing μ to appear in only one term of each equation:

$$\begin{aligned}
\left(\mu - \frac{\Delta t}{2} K_{00}\right) u_0 - \Delta t K_{01} u_1 - \Delta t K_{02} u_2 - \cdots - \frac{\Delta t}{2} K_{0n} u_n &= 0 \\
-\frac{\Delta t}{2} K_{10} u_0 + (\mu - \Delta t K_{11})\, u_1 - \Delta t K_{12} u_2 - \cdots - \frac{\Delta t}{2} K_{1n} u_n &= 0 \\
\vdots \qquad & \vdots \\
-\frac{\Delta t}{2} K_{n0} u_0 - \Delta t K_{n1} u_1 - \cdots + \left(\mu - \frac{\Delta t}{2} K_{nn}\right) u_n &= 0.
\end{aligned}$$

Again, the equations may be written in matrix notation as

$$K_H U = 0$$

where

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

U is the same as the nonhomogeneous case:

$$U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix},$$

and K_H is the coefficient matrix for the homogeneous equations

$$K_H = \begin{bmatrix} \mu - \frac{\Delta t}{2} K_{00} & -\Delta t K_{01} & \cdots & -\frac{\Delta t}{2} K_{0n} \\ -\frac{\Delta t}{2} K_{10} & \mu - \Delta t K_{11} & \cdots & -\frac{\Delta t}{2} K_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\Delta t}{2} K_{n0} & -\Delta t K_{n1} & \cdots & \mu - \frac{\Delta t}{2} K_{nn} \end{bmatrix}.$$

An infinite number of nontrivial solutions exist iff $|K_H| = 0$. This condition can be used to find the eigenvalues λ of the system by finding $\mu = \frac{1}{\lambda}$ as the zeros of $|K_H| = 0$.

Example: Use the trapezoid rule to find a numeric solution to the homogeneous Fredholm equation

$$u(x) = \lambda \int_0^1 K(x,t) u(t)\, dt \tag{29}$$

with the symmetric kernel

$$K(x,t) = \begin{cases} x(1-t), & 0 \le x \le t \\ t(1-x), & t \le x \le 1 \end{cases}$$

at $x = 0$, $x = \frac{1}{2}$, and $x = 1$. Therefore $n = 2$ and $\Delta t = \frac{1}{2}$, and

$$K_{ij} = K\left(\frac{i}{2}, \frac{j}{2}\right), i, j = 0, 1, 2.$$

The resulting matrix is

$$\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu - \frac{1}{8} & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

For the system of homogeneous equations obtained to have a nontrivial solution, the determinant must be equal to zero:

$$\begin{vmatrix} \mu & 0 & 0 \\ 0 & \mu - \frac{1}{8} & 0 \\ 0 & 0 & \mu \end{vmatrix} = \mu^2 \left(\mu - \frac{1}{8}\right) = 0,$$

or, $\mu = 0$ or $\mu = \frac{1}{8}$.

For $\mu = \frac{1}{8}$, $\lambda = \frac{1}{\mu} = 8$ and substituting into the system of equations obtains $u_0 = 0$, $u_2 = 0$, and u_1 as an arbitrary constant. Therefore there are two zeros at $x = 0$ and $x = 1$, but an arbitrary value at $x = \frac{1}{2}$. It can be found by approximating the function $u(x)$ as a triangular function connecting the three points $(0, 0)$, $(\frac{1}{2}, u_1)$, and $(1, 0)$:

$$u(x) = \begin{cases} 2u_1 x, & 0 \le x \le \frac{1}{2} \\ -2u_1(x-1), & \frac{1}{2} \le x \le 1 \end{cases},$$

and making its norm be one:

$$\int_0^1 u^2(x)\, dx = 1.$$

Substituting results in

$$\begin{aligned} 4u_1^2 \int_0^{\frac{1}{2}} x^2\, dx + 4u_1^2 \int_{\frac{1}{2}}^1 (x-1)^2\, dx &= 1 \\ \frac{u_1^2}{6} + \frac{u_1^2}{6} &= 1 \\ \frac{u_1^2}{3} &= 1 \\ u_1 &= \sqrt{3}. \end{aligned}$$

So the approximate numerical values are $u(0) = 0$, $u(\frac{1}{2}) = \sqrt{3}$, and $u(1) = 0$.

For comparison with an exact solution, the orthonormal eigenfunction is chosen to be $u_1(x) = \sqrt{2}\sin \pi x$, which corresponds to the eigenvalue $\lambda_1 = \pi^2$, which is close to the approximate eigenvalue $\lambda = 8$. The exact solution gives $u_1(0) = \sqrt{2}\sin 0 = 0$, $u_1(\frac{1}{2}) = \sqrt{2}\sin\frac{\pi}{2} \approx 1.4142$, and $u_1(1) = \sqrt{2}\sin\pi = 0$.

7 An example of a separable kernel equation

Solve the integral equation

$$\phi(x) - \lambda \int_0^{\pi} \sin(x+t)\phi(t)dt = 1 \tag{30}$$

where $\lambda \neq \pm\dfrac{2}{\pi}$.

The kernel is $K(x,t) = \sin x \cos x + \sin t \cos x$ where

$$\begin{aligned} a_1(x) &= \sin x \\ a_2(x) &= \cos x \\ b_1(x) &= \cos t \\ b_2(x) &= \sin t \end{aligned}$$

where we have use the additive identity $\sin(x+t) = \sin x \cos t + \sin t \cos x$.

Choosing $f(x) = 1$ we obtain

$$\begin{aligned} f_1 &= \int_0^\pi b_1(t) f(t) dt = \int_0^\pi \cos t dt = 0 \\ f_2 &= \int_0^\pi b_2(t) f(t) dt = \int_0^\pi \sin t dt = 2 \end{aligned}$$

and so

$$F = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{31}$$

$$\begin{aligned} a_{11} &= \int_0^\pi b_1(t) a_1(t) dt = \int_0^\pi \sin t \cos t dt = 0 \\ a_{12} &= \int_0^\pi b_1(t) a_2(t) dt = \int_0^\pi \cos^2 t dt = \frac{\pi}{2} \\ a_{21} &= \int_0^\pi b_2(t) a_1(t) dt = \int_0^\pi \sin^2 t dt = \frac{\pi}{2} \\ a_{12} &= \int_0^\pi b_2(t) a_2(t) dt = \int_0^\pi \sin t \cos t dt = 0 \end{aligned}$$

which yields

$$A = \begin{bmatrix} 0 & \frac{\pi}{2} \\ \frac{\pi}{2} & 0 \end{bmatrix} \tag{32}$$

Now we can rewrite the original integral equation as $C = F + \lambda AC$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 0 & \frac{\pi}{2} \\ \frac{\pi}{2} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{33}$$

or

$$\begin{bmatrix} 1 & \frac{\pi}{2}\lambda \\ \frac{\pi}{2}\lambda & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{34}$$

$$\begin{aligned} c_1 - \frac{\lambda\pi}{2} c_2 &= 0 \\ c_2 - \frac{\lambda\pi}{2} c_1 &= 2\lambda \end{aligned}$$

with solutions $c_1 = 8\lambda/(4-\lambda^2\pi^2), c_2 = 4\lambda^2\pi/(4-\lambda^2\pi^2)$.

The general solution is $\phi(x) = 1 + c_1 a_2 + c_2 a_1 = 1 + c_1 \cos x + c_2 \sin x$ and the unique solution of the integral equation is $\phi(x) = 1 + \dfrac{8\lambda}{(4-\lambda^2\pi^2)} \cos x + \dfrac{4\lambda^2\pi}{(4-\lambda^2\pi^2)} \sin x$.

IVP1

$$\begin{aligned} &\phi'(x) = F(x, \phi(x)), 0 \le x \le 1 \\ &\phi(0) = \phi_0 \\ \\ &\Rightarrow \phi(x) = \int_0^x F(t, \phi(t))dt + \phi_0 \end{aligned}$$

IVP2

$$\begin{aligned} &\phi''(x) = F(x, \phi(x)), 0 \le x \le 1 \\ &\phi(0) = \phi_0, \phi'(0) = \phi_0' \\ \\ &\Rightarrow \phi'(x) = \int_0^x F(t, \phi(t))dt + \phi_0' \\ &\Rightarrow \phi(x) = \int_0^x ds \int_0^s F(t, \phi(t))dt + \phi_0'(x) + \phi_0 \end{aligned}$$

we know that:

$$\begin{aligned} \int_0^x ds \int_0^s G(s,t)dt &= \int_0^x dt \int_t^x G(s,t)ds \\ \int_0^x ds \int_0^s F(t, \phi(t))dt &= \int_0^x (x-t)\ F(t, \phi(t))dt \end{aligned}$$

Therefore,

$$\Rightarrow \phi(x) = \int_0^x (x-t)\ F(t,\phi(t))dt + \phi_0'(x) + \phi_0$$

Theorem 1 Degenerate Kernels *If $k(x,t)$ is a degenerate kernel on $[a,b]\times[a,b]$ which can be written in the form $k(x,t) = \sum_{i=0}^{n} a_i(x)b_i(t)$ where $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ are continuous, then*

1. *for every continuous function f on $[a,b]$ the integral equation*

$$\phi(x) - \lambda \int_a^b k(x,t)\ \phi(t)dt = f(x)\ (a \le x \le b) \tag{35}$$

possesses a unique continuous solution ϕ, or

2. *the homogeneous equation*

$$\phi(x) - \lambda \int_a^b k(x,t)\ \phi(t)dt = 0\ (a \le x \le b) \tag{36}$$

has a non-trivial solution ϕ, in which case eqn. 35 will have non-unique solutions if and only if $\int_a^b f(t)\overline{\psi(t)}\ dt = 0$ for every continuous solution ψ of the equation

$$\psi(x) - \bar{\lambda} \int_a^b \overline{k(t,x)}\ \psi(t)dt = 0\ (a \le x \le b). \tag{37}$$

Theorem 2 *Let K be a bounded linear map from $L_2(a,b)$ to itself with the property that $\{K\phi : \phi \in L_2(a,b)\}$ has finite dimension. Then*

1. *the linear map $I - \lambda K$ has an inverse and $(I-\lambda K)^{-1}$ is a bounded linear map, or*
2. *the equation $\psi - \lambda K\psi = 0$ has a non-trivial solution $\psi \in L_2(a,b)$.*

In the specific case where the kernel k generates a compact operator on $L_2(a,b)$, the integral equation

$$\phi(x) - \lambda \int_a^b k(x,t)\ \phi(t)dt = f(x)\ (a \le x \le b)$$

will have a unique solution, and the solution operator $(I-\lambda K)^{-1}$ will be bounded, provided that the corresponding homogeneous equation

$$\phi(x) - \lambda \int_a^b k(x,t)\ \phi(t)dt = 0\ (a \le x \le b)$$

has only the trivial solution $L_2(a,b)$.

8 Bibliographical Comments

In this last section we provide a list of bibliographical references that we found useful in writing this survey and that can also be used for further studying Integral Equations. The list contains mainly textbooks and research monographs and do not claim by any means that it is complete or exhaustive. It merely reflects our own personal interests in the vast subject of Integral Equations. The books are presented using the alphabetical order of the first author.

The book [Cor91] presents the theory of Volterra integral equations and their applications, convolution equations, a historical account of the theory of integral equations, fixed point theorems, operators on function spaces, Hammerstein equations and Volterra equations in abstract Banach spaces, applications in coagulation processes, optimal control of processes governed by Volterra equations, and stability of nuclear reactors.

The book [Hac95] starts with an introductory chapter gathering basic facts from analysis, functional analysis, and numerical mathematics. Then the book discusses Volterra integral equations, Fredholm integral equations of the second kind, discretizations, kernel approximation, the collocation method, the Galerkin method, the Nystrom method, multi-grid methods for solving systems arising from integral equations of the second kind, Abel's integral equation, Cauchy's integral equation, analytical properties of the boundary integral equation method, numerical integration and the panel clustering method and the boundary element method.

The book [Hoc89] discusses the contraction mapping principle, the theory of compact operators, the Fredholm alternative, ordinary differential operators and their study via compact integral operators, as well as the necessary background in linear algebra and Hilbert space theory. Some applications to boundary value problems are also discussed. It also contains a complete treatment of numerous transform techniques such as Fourier, Laplace, Mellin and Hankel, a discussion of the projection method, the classical Fredholm techniques, integral operators with positive kernels and the Schauder fixed-point theorem.

The book [Jer99] contains a large number of worked out examples, which include the approximate numerical solutions of Volterra and Fredholm, integral equations of the first and second kinds, population dynamics, of mortality of equipment, the tautochrone problem, the reduction of initial value problems to Volterra equations and the reduction of boundary value problems for ordinary differential equations and the Schrodinger equation in three dimensions to Fredholm equations. It also deals with the solution of Volterra equations by using the resolvent kernel, by iteration and by Laplace transforms, Green's function in one and two dimensions and with Sturm-Liouville operators and their role in formulating integral equations, the Fredholm equation, linear integral equations with a degenerate kernel, equations with symmetric kernels and the Hilbert-Schmidt theorem. The book ends with the Banach fixed point in a metric space and its application to the solution of linear and nonlinear integral equations and the use

of higher quadrature rules for the approximate solution of linear integral equations.

The book [KK74] discusses the numerical solution of Integral Equations via the method of invariant imbedding. The integral equation is converted into a system of differential equations. For general kernels, resolvents and numerical quadratures are being employed.

The book [Kon91] presents a unified general theory of integral equations, attempting to unify the theories of Picard, Volterra, Fredholm, and Hilbert-Schmidt. It also features applications from a wide variety of different areas. It contains a classification of integral equations, a description of methods of solution of integral equations, theories of special integral equations, symmetric kernels, singular integral equations, singular kernels and nonlinear integral equations.

The book [Moi77] is an introductory text on integral equations. It describes the classification of integral equations, the connection with differential equations, integral equations of convolution type, the method of successive approximation, singular kernels, the resolvent, Fredholm theory, Hilbert-Schmidt theory and the necessary background from Hilbert spaces.

The handbook [PM08] contains over 2500 tabulated linear and nonlinear integral equations and their exact solutions, outlines exact, approximate analytical, and numerical methods for solving integral equations and illustrates the application of the methods with numerous examples. It discusses equations that arise in elasticity, plasticity, heat and mass transfer, hydrodynamics, chemical engineering, and other areas. This is the second edition of the original handbook published in 1998.

The book [PS90] contains a treatment of a classification and examples of integral equations, second order ordinary differential equations and integral equations, integral equations of the second kind, compact operators, spectra of compact self-adjoint operators, positive operators, approximation methods for eigenvalues and eigenvectors of self-adjoint operators, approximation methods for inhomogeneous integral equations, and singular integral equations.

The book [Smi58] features a carefully chosen selection of topics on Integral Equations using complex-valued functions of a real variable and notations of operator theory. The usual treatment of Neumann series and the analogue of the Hilbert-Schmidt expansion theorem are generalized. Kernels of finite rank, or degenerate kernels, are discussed. Using the method of polynomial approximation for two variables, the author writes the general kernel as a sum of a kernel of finite rank and a kernel of small norm. The usual results of orthogonalization of the square integrable functions of one and two variables are given, with a treatment of the Riesz-Fischer theorem, as a preparation for the classical theory and the application to the Hermitian kernels.

The book [Tri85] contains a study of Volterra equations, with applications to differential equa-

tions, an exposition of classical Fredholm and Hilbert-Schmidt theory, a discussion of orthonormal sequences and the Hilbert-Schmidt theory for symmetric kernels, a description of the Ritz method, a discussion of positive kernels and Mercer's theorem, the application of integral equations to Sturm-Liouville theory, a presentation of singular integral equations and non-linear integral equations.

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