CRITIQUES & CONTENTIONS

History of Ancient Mathematics

SOME REFLECTIONS ON THE STATE OF THE ART

*By Sabetai Unguru**

Denn eben wo Begriffe fehlen
Da stellt ein Wort zur rechten Zeit sich ein.
—JOHANN WOLFGANG VON GOETHE[1]

THE HISTORY OF MATHEMATICS typically has been written as if to illustrate the adage "anachronism is no vice." Most contemporary historians of mathematics, being mathematicians by training, assume tacitly or explicitly that mathematical entities reside in the world of Platonic ideas where they wait patiently to be discovered by the genius of the working mathematician. Mathematical concepts, constructive as well as computational, are seen as eternal, unchanging, unaffected by the idiosyncratic features of the culture in which they appear, each one clearly identifiable in its various historical occurrences, since these occurrences represent different clothings of the same Platonic hypostasis.

Various forms of the same mathematical concept or operation are not considered merely mathematically equivalent but also historically equivalent. Indeed mathematical equivalence is taken to represent historical equivalence. Since the mathematical Forms are eternal and since in their works mathematicians of all ages share in the expression of the same Forms, the specific mathematical idiom used by a mathematician has no bearing on the content of his thought. Mathematical language is at best a secondary appurtenance of the mathematical culture of any epoch. The mathematical kernel is untouched by the peculiar language used, since all mathematical languages lead back to the same ideal Forms. This makes the various casts in which the same mathematical truth has been expressed throughout the centuries completely equivalent. As one of my colleagues put it: "Under such an ontology, the object of the history of mathematics becomes the task of identifying the ideal forms present in the work of each historical author and apportioning out proper credit to that mathematician who first gave expression to one of these eternal forms, i.e., who first brought it out of the eternal Platonic realm into the world of human consciousness." This is precisely the task performed traditionally by the historian of mathematics.

But if scholars continue to neglect the peculiar specificities of a given mathematical

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[1] *Faust* I (Mephistopheles speaks).

culture, whether as a result of explicitly stated or implicitly taken-for-granted assumptions, then by definition their work is ahistorical and should be recognized as such by the community of historians. History, as Aristotle knew, focuses on the idiosyncratic rather than the nomothetic.[2] It is impossible for modern man to think like an ancient Greek. Historical understanding, however, involves the attempt at faithful reconstruction of the past. In intellectual history this necessarily means the avoidance of conceptual pitfalls and interpretive anachronisms. Though it is impossible to think like Euclid, it is rather facile to think obtrusively unlike him. We cannot know what went through Euclid's mind when he wrote the *Elements*. But we can determine what Euclid could not have thought when he compiled his great work. He, most likely, did not employ concepts or operations for which there is no genuine evidence either in his time or in the works of his predecessors. This much is safe to conclude. Furthermore, he clearly could not have foreseen what mathematicians and historians of mathematics were going to do in the long run to his *Elements*; he could not have used mathematical devices and procedures which were invented many hundreds of years after his death. This much is obvious too. Given that we cannot think like Euclid, we should, nevertheless, strive to avoid thinking unlike him when elucidating and commenting on his writings. This is (and must remain) the historian's goal. One way of thinking *unlike* Euclid is to use the algebraic approach in interpreting his works.

It continues to be habitual among some historians of mathematics to say that what really lies behind Euclid's geometrically couched statements are algebraic reasonings, appearing in geometrical garb because of the lack of an appropriate algebraic symbolism. This strikes me as both inaccurate and unilluminating. To see this, let us ask the following question: how illuminating would it be to propose that Euclid really thought in Sanskrit but because of his ignorance of the Sanskrit alphabet, had to use the Greek one and consequently expressed himself in Greek? Greek mathematics must be understood in its own right. This can be done by refusing to apply to its analysis foreign, anachronistic criteria. The only acceptable meta-language for a historically sympathetic investigation and comprehension of Greek mathematics seems to be ordinary language, not algebra.

However, many scholars and in particular B. L. van der Waerden and Hans Freudenthal do not endorse these ideas.[3] Instead, they argue that mathematicians can easily discern in Greek mathematics its underlying algebraic basis and interpret it accordingly. Who would deny this manifest truth? The real question is: how accurate and factual is the mathematicians' interpretation? Is the treatment to which mathematicians have submitted Greek mathematics historically adequate? In fact, van der Waerden and Freudenthal argue the cogency and the completeness of the mathematicians' interpretation and treatment of Greek mathematics in algebraic form. But there is a serious problem here. The fact that modern mathematicians can interpret Greek mathematics algebraically is one thing. The conclusion that therefore the train of thought of the Greek mathematicians was algebraic is an entirely different matter. The step from the former statement to the latter is both a logical and a historical *non sequitur*. "Logical," for obvious reasons; "historical," because it is possible to show

[2]Aristotle, *Poetica* 1451b 1–19.

[3]B. L. van der Waerden, "Defence of a 'Shocking' Point of View," *Archive for History of Exact Sciences*, 1975, *15*:199–210 and Hans Freudenthal, "What Is Algebra and What Has It Been in History," *Arch. Hist. Exact Sci.*, 1977, *16*:189–200, both written in response to my article "On the Need To Rewrite the History of Greek Mathematics," *Arch. Hist. Exact Sci.*, 1975, *15*:67–114.

by an analysis of Greek mathematical texts that the assumption of an underlying algebraic foundation for Greek mathematics leads to insoluble dilemmas and dreadful quandaries.[4]

In brief, van der Waerden defines algebra as *"the art of handling algebraic expressions like* $(a + b)^2$ *and of solving equations like* $x^2 + ax = b$*."*[5] But no algebra exists in Babylonian and pre-Diophantian Greek mathematical sources. "Babylonian and Greek algebra" came into being only after the specific, numerical Babylonian examples and the Greek geometrical propositions had been transcribed into algebraic language; only as a result of the mathematician's elucidation of the texts was "algebra" brought into existence.[6] But the text itself did not present this elucidation; the imaginative creation of the interpreter did so. And there is a mathematical imagination and a historical imagination, and they typically run on different tracks. Finally, continuing allegiance to the algebraic interpretation of Babylonian mathematics is rendered still more untenable by the following rather damaging confession: "In den eigentlich mathematischen Texten . . . die meisten Beispiele sicherlich von ihrem Resultat aus hergerichtet sind."[7] If this is true (and it seems to be, for most answers are "nice," even numbers), then what is the basis for the claim that the Babylonians solved equations? Do we as a rule solve equations by starting with the answer?

Those who perceive an algebraic substructure bolstering Greek mathematics claim that the Greeks started with algebraic problems but, then, translated them into a geometric format. They have reached this conclusion, according to van der Waerden, by studying "the wording of the theorems" and by trying "to reconstruct the original ideas of the author. We found it *evident* that these theorems did not arise out of geometrical problems [!]. We were not able to find any interesting geometrical problem that would give rise to theorems like II 1–4. On the other hand, we found that the explanation of these theorems as arising from algebra worked well. Therefore we adopted the latter explanation."[8]

But what evidence does van der Waerden present to demonstrate that "these theorems did not arise out of geometrical problems"? The answer, he tells us, is that no "interesting geometrical problem" leads to them. How does he know? Answer: he could not find any. But the conclusion is unwarranted, since even if it is true that no interesting geometrical problems led to them, it does not follow that noninteresting geometrical problems did not lead to them either. Furthermore, what *is* an interesting geometrical problem? Van der Waerden does not say, but the answer is implicit in what follows: "we found that the explanation of these theorems as arising from algebra worked well. Therefore. . . ." An interesting geometrical problem, then, seems to be a problem the assumption of which "works well" in explaining the origin of the theorems under discussion. And van der Waerden has decided arbitrarily (since he could not check all possible geometrical theorems and problems) that there are no

[4]See "On the Need to Rewrite."

[5]"Defence," p. 199.

[6]In this context, S. Gandz, who advances his own algebraic interpretation of ancient mathematics in "The Origin and Development of the Quadratic Equations in Babylonian, Greek and early Arabic Algebra," *Osiris,* 1938, *3*:405–557, says the following: "The commentary of NEUGEBAUER . . . has no foundation in the text. It only shows how far away from the truth we may err, if we try, by all means, to detect our modern school formulas in the old Babylonian text" (p. 423).

[7]Otto Neugebauer, *Vorgriechische Mathematik* (Berlin: Springer-Verlag, 1934; reprinted Springer-Verlag, 1969), p. 33.

[8]"Defence," pp. 203–204.

"interesting geometrical problems" working well under the circumstances. On the other hand, what works well is the assumption of an underlying algebraic foundation to Greek geometry. What does "working well" mean, then? Again, no answer is provided, but it would clearly seem to mean something removing difficulties and enabling one to cut through to the root and thus come up with "simple," "convincing," straightforward explanations. Ultimately, then, in the paragraph under discussion, van der Waerden does say that Greek geometry (at least some important parts of it) is, taken by itself, unfathomable, puzzling, weird, and that one can get rid of these unsavory features by assuming a hidden algebraic basis to it. Therefore, "the Greeks started with algebraic problems and translated them into geometric language," Q.E.D.[9]

Leaving aside the circularity of the entire argument, and the conflation of logic and history that it involves, van der Waerden's assertions represent an unconscious but nevertheless clear-cut vindication of the argument that the real roots of the methodological position embodied in the concept "geometric algebra" lie in the modern mathematician's ability to read geometric texts algebraically without any historical qualms.

II

Why did the Greeks, according to the proponents of the idea, disguise algebra in geometrical garb? Freudenthal gives three different answers: one historical, the second philosophical (its pertinence entirely escapes me), and the third "traditional." His "historical" answer speaks of a "torturous path through foundations of mathematics,"[10] which came to an end with the Eudoxian theory of proportions. But since there was neither genuine foundational work nor a real *Grundlagenkrisis* (as Hasse and Scholz and Van der Waerden referred to it[11]) in pre-Eudoxian times,[12] speaking of "the Greek end of the torturous path through foundations of mathematics, EUDOXOS'

[9]It is this very same approach which is involved in identifying the purely geometrical problem of the application of areas as the Greek method of solution of quadratic equations, later equated with the Babylonian method: the modern mathematician can indeed translate Greek geometry and Babylonian specific-number manipulations into the algebraic language. Thus the parabolic application of areas, in which one is asked to apply to a given straight line a rectangle *equal* to a given square, can be transcribed as $ax = b^2$, if the given line is a and the given square b^2; to apply to the given line a rectangle equal to the given square such that the applied rectangle *falls short* of the second extremity of the given line by a square (the elliptical application of areas) can be transcribed as $x + y = a$, $xy = b^2$; and, to apply to the given line a rectangle equal to the given square such that the applied rectangle *exceeds* the second extremity of the given line by a square (the hyperbolic application of areas) can be transcribed as $x - y = a$, $xy = b^2$. This mathematical possibility, however, is not a satisfactory historical justification for the claimed identity of the Greek and the algebraic procedure. Moreover, strictly speaking, it is not the case that Euclid, *Elements* I.44 corresponds exactly to the simple parabolic application of areas. The simple parabolic application does not lead (as we saw) to a quadratic equation. If anything, it corresponds to the division of a given product (area) by a given magnitude (line). Only *Elements* VI.28 and 29 lead, when transcribed algebraically, to complete quadratic equations corresponding respectively to elliptical and hyperbolic application of areas. In this context, see A. Szabo, "Zum Problem der sog. 'Geometrischen Algebra' in Euklids Elementen," completed in 1975 for a *Festschrift* in honor of Willy Hartner, p. 7 of the preprint.

[10]Freudenthal, "What Is Algebra," p. 191.

[11]H. Hasse and H. Scholz, "Die Grundlagenkrisis der griechischen Mathematik," *Kant Studien,* 1928, *33*:4–34; B. L. van der Waerden, "Zenon und die Grundlagenkrise der griechischen Mathematik," *Mathematische Annalen,* 1940–1941, *117*:141–161.

[12]Wilbur R. Knorr, *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry* (Dordrecht/Boston: D. Reidel, 1975), pp. 40–42, 50, 305–313; cf. also Hans Freudenthal, "Y avait-il une crise des fondements des mathématiques dans l'antiquite?" *Bulletin de la Societé Mathématique de Belgique,* 1966, *8*:43-55.

theory" is misleading and consequently Freudenthal's so-called historical answer is nonhistorical and a nonanswer. The other answers do not fare any better. For example, in the "traditional" answer Freudenthal points out that "Once canonised, the *Elements* were sacrosanct . . . The mathematical community was small. To be understood within it, you had to quote EUCLID and to speak his language."[13] Fine, but why did Euclid, then, adopt the very same language? In sum, Freudenthal's "three main reasons" for the alleged disguise by the Greeks of algebra under the cloak of geometry are not good reasons.

"The efficiency of a symbolism is determined by the ease with which the user can move within it, by the algorithmic autonomy it provides."[14] Granted. But to judge from the texts, neither the Babylonian nor the ancient Greek moved at all "within it," and it possessed for them no "algorithmic autonomy" whatever, since it was not yet born. "It is the virtue of symbolism that it allows us for most of the time, rather than identify object and symbol . . . to forget about what the symbol means." But this task is indeed impossible in Babylonian and Greek mathematical texts, where the object is always identified, either as specific numbers or as spacial diagrams. It is impossible to forget that "14, 30" means exactly "14, 30" and "line *AB*" means exactly "line *AB*"! It is simply not true that "Almost never in the *Elements* or anywhere else in Greek mathematics does *AB* mean a line or a line segment."[15]

As Freudenthal would have it, *Elements* V is "algebra and nothing else"; it is, moreover, "a general theory of magnitude . . . independent of dimension or any characteristic of specific magnitudes."[16] The problem with such a characterization is the existence of *Elements* VII, in which many of the things dealt with in Book V are repeated and applied specifically to numbers (integers). In the presence of a general theory of magnitude, such a procedure would not have been just repetitious and superfluous but outright senseless. Numbers, after all, are specific instances of magnitude, and what is true of magnitudes in general is also true of numbers. In writing Book VII, then, Euclid did not simply follow tradition (as Heath thinks),[17] thereby merely proving over again for numbers propositions which he already proved for all magnitudes, including numbers. Book V, it seems, presents a general theory of proportion applicable to all kinds of magnitudes but not to numbers. The reason for this is that numbers for the Greeks are not instances of a concept of general magnitude. "Magnitude, in fact, corresponds to one of the two divisions of *quantity*, *ποσόν*, namely the continuous (as a line, a surface, or a body) whereas a number is *discrete*."[18] Numbers (integers) are not illustrations of something else, they are entities in their own right, with their own distinctive features, definitions, and so forth. This is what enables Wilbur Knorr to say that "Book VII does not merely duplicate Book V. It develops a body of analogous material for *the separate class of integers* [my emphasis]; that is, it is *required* as an independent treatment, not a duplication of a special case of Book V."[19]

[13] "What Is Algebra," p. 191. Freudenthal's philosophical answer reads: "Though in daily use by laymen as well as mathematicians, fractions were taboo in highbrow mathematics, because philosophy forbade the division of the unit" (*ibid.*).

[14] *Ibid.*, p. 192.

[15] *Ibid.*

[16] *Ibid.*, p. 193.

[17] T. L. Heath, *The Thirteen Books of Euclid's Elements*, 3 vols. (Cambridge: Cambridge University Press, 1908), Vol. II, p. 113.

[18] T. L. Heath, *Mathematics in Aristotle* (Oxford: Clarendon Press, 1949), p. 45.

[19] Knorr, *Evolution of the Euclidean Elements*, p. 309.

Why should this be so? Basically, because of the Greek view that arithmetic is an independent, not a derivative, discipline and that geometry and arithmetic are different genera having their own domains, disposing of their own techniques of demonstration, and dealing with their own subject matter. Pursuing them properly means refraining from infringing upon the territory of one by means of the tools and methods of the other.[20]

If I had been aware of the existence of Euclid's *Data,* argues Freudenthal, I "would never have claimed there were no equations in Greek geometry." For Freudenthal, the *Data* is a "textbook on solving equations." He summarizes the ninety-four propositions contained therein in a succinctly and strikingly epigrammatic statement: "Given certain magnitudes *a, b, c* and a relation *F(a, b, c, x),* then *x,* too, is given . . . ."[21] But the fact remains that Greek geometry contained no equations. One cannot find even one equation in the entire text of the *Data.* Proof (as the Hindu mathematician would say): "Look!" Unless one has at his disposal the algebraic language and the capacity to translate into it, it is impossible to sum up this little treatise of rather varied content as offhandedly as Freudenthal has done. Indeed, had Euclid at his disposal Freudenthal's functional notation, it is rather easy to infer that he would not have needed ninety-four propositions to get his point across.

Each case in Euclid's *Data* is unique, having its own method of analysis, and none is subsumable under or reducible to other cases. "Datarum magnitudinum ratio inter se data est" (Prop. I) and "Si data magnitudo ad aliam magnitudinem rationem habet datam, data est etiam illa magnitudine" (Prop. II)—to use perhaps the simplest illustration possible—are not for Euclid both instances of "Given *a, b, c,* and $y = F(a, b, c, x)$, *x* is also given," but two different problems, interesting in their own right, having their own solutions. Of course, Freudenthal's description is mathematically correct. Historically, however, it is wanting. Heath is much more to the point when he says: "The *Data* . . . are still concerned with *elementary geometry,* though forming part of the introduction to higher analysis. Their form is that of propositions proving that, if certain things *in a figure* are given (in magnitude, in species, etc.), something else is given. The subject-matter is much the same as that of the planimetrical books of the *Elements,* to which the *Data* are often supplementary."[22]

This is what the *Data* is, not a textbook on solving equations, but a treatise presenting another approach to elementary geometry (other than that of the *Elements,* that is). Neither are Archimedes' works instances of "algebraic procedure in Greek mathematics."[23] Heath's edition is "in modern notation."[24] It is faithful only to the disembodied mathematical content of the Archimedean text, but not to its form. And this is crucial. If one abandons Archimedes' form and transcribes his rhetorical statements by means of algebraic symbols, manipulating and transforming the latter, then clearly "the algebraic procedure" appears. But this procedure itself is not "in Greek mathematics." It is a result (as Freudenthal himself states it) of "replacing vernacular by artificial language, and numbering variables by cardinals, a quite recent mathematical tool."[25] Indeed! Archimedes' text is anchored securely in the *terra firma* of Greek geometry. If one is not willing to compress wording, to replace

[20] Aristotle, *Posterior Analytics* 75a37–75b20.

[21] "What Is Algebra," p. 194. By the way, references to the *Data* appear in my paper in a number of places, e.g., p. 81, n. 26, and p. 108, n. 106.

[22] Heath, *Elements,* Vol. I, p. 8, my italics.

[23] "What Is Algebra," p. 195.

[24] T. L. Heath, *The Works of Archimedes* (Cambridge: Cambridge University Press, 1897), title page.

[25] "What Is Algebra," p. 196.

"vernacular" by artificial language, to introduce variables and number them by cardinals, and to apply all the other technical tricks which are "quite recent mathematical tools," then Archimedes' proof of Proposition X of ΠΕΡΙ ΕΛΙΚΩΝ is geometric, not algebraic. This was discerned in a curious way even by Heath, who justified his algebraic procedure and the use of the symbols $A_1, A_2, \ldots A_n$, "in order to exhibit *the geometrical character of the proof*."[26]

Dijksterhuis himself in his *Archimedes* said: "in a representation of Greek proofs in the symbolism of modern algebra it is often precisely the most characteristic qualities of the classical argument which are lost, so that the reader is not sufficiently obliged to enter into the train of thought of the original. . . ."[27]

III

Both Freudenthal and van der Waerden have constructed identical operative definitions of algebra, thereby creating significant problems in their analyses of Greek geometry. Freudenthal says: "This ability to describe relations and solving procedures, and the techniques involved *in a general way*, is in my view of algebra such an important feature of algebraic thinking that I am willing to extend the name 'algebra' to it. . . . But what is in a name?"[28] However, it is precisely the inability of the Babylonian mathematician "to describe relations and solving procedures, and the techniques involved *in a general way*" that warrants his disqualification as algebraist. What the Babylonian mathematician lacks is precisely the ability to dispense with specific, definite numbers, and it is this deficiency that dictates the particular form of his approach. What he can produce is recipes, not general formulas.

With respect to the Greek mathematician (geometer), on the other hand, though it is legitimate to see his approach as a general approach (the so-called theorem of Pythagoras is true of *any* right-angled triangle, etc.), the language he uses is the geometric language and the generality involved is an outgrowth of dealing with geometrical and *not* with algebraic entities. Consequently, by Freudenthal's own criteria of "algebraic thinking," Babylonian and Greek mathematics are nonalgebraic.

"What's in a name?" asks Freudenthal and uses the question even as a motto for his article. The answer, clearly enough, is "it depends." Names are words, and words are important when used thoughtfully. As a matter of fact, it is possible to argue that all there is is, one way or another, in words. The *Iliad* and *Hamlet* are in words; and so is the *Magna Charta*. The Bible is in words; and so is the American Declaration of Independence. All of mathematics is in a very definite sense in words. Thought and feeling (beyond inarticulate physiological reactions) are in words. Artistic experience is in a proper sense in words, for no informed, thoughtful reaction to and communication about a work of art is possible in the absence of articulate expression, which is again in words. Our meaningful access to reality (whatever it may consist of) is always mediate: we know the world through words.[29]

[26] Heath, *Archimedes*, p. 109, my italics.

[27] E. J. Dijksterhuis, *Archimedes* (Copenhagen: Munksgaard, 1956), p. 7.

[28] "What Is Algebra," pp. 193–194, my italics.

[29] In a pregnant way, the Venerable Inceptor expressed this view as follows: "Si dicas: nolo loqui de vocibus sed tantum de rebus, dico quod quamvis velis loqui tantum de rebus, tamen hoc non est possibile nisi mediantibus vocibus vel conceptibus vel aliis signis" (William of Ockham, *Commentary on the Sentences*, I, dist. 2, quest. 1, in *Super quatuor libros sententiarum (In sententiarum I)*, being Vol. III (1495) of Guillelmus de Occam, O.F.M., *Opera plurima* (Lyons, 1494–1496; reprinted London: Gregg Press, 1962, in 4 vols.).

But words can be misused. *Mein Kampf* and the *Protocols of the Elders of Zion* are also in words; and so is *The National Enquirer.* Words are powerful weapons, and men are governed (or misgoverned) with, by, and in words. And so, "what's in a word?" As always, "it all depends." Is the word used carefully? Does the user follow the advice of Paul to the Ephesians: "Let no man deceive . . . with vain words"? Are words used and understood pertinently with reference to the subject matter, according to the old legal maxim, "Verba accipienda sunt secundum subjectam materiam?" If the answer to the above questions is positive, then there is a lot in a word; if negative, the word is misleading and therefore dangerous.

The use of the word "algebra" as a term descriptive of Babylonian and Greek mathematics is a misuse of the word. When the questions enumerated above are asked in connection with that use, all the answers come out negative. The word "algebra" is used carelessly; its use is deceiving since it leads to a translation of ancient mathematical texts into a historically inappropriate language; and, if "algebra" has its proper meaning, the use of the term is unsuited to the subject matter. Words are judgments, or, as Nietzsche put it, preconceived judgments; and this is how it should be. But some judgments carry conviction while others are blatantly unjust. The word "algebra" in the context discussed belongs to the latter category.

Enthusiasts of algebraic interpretations of Greek geometry have violated one of the fundamental tenets of historical scholarship. History is the study of the present traces of past events from the standpoint of change and the particular, the idiosyncratic.[30] Although long-lasting structures, stable frameworks, and durable, quasi-constant features are legitimate topics of historical investigation, they are not what makes history what it is.[31] History is primarily, essentially interested in the event *qua* particular event, in the specific happening, in change from an identifiable, individual characteristic to another identifiable, individual characteristic. History is not (or is primarily not) striving to bunch events together, to crowd them under the same heading by draining them of their individualities. On the contrary, history is the attempt at understanding each past event in its own right. The domain of history, then, is the idiosyncratic.

The historian of ideas does not discharge his obligation by showing merely the extent to which past ideas are like modern ideas. His main effort should be in the direction of showing the extent to which past ideas were unlike modern ones, irrespective of the fact that they might (or might not) have led to the modern ideas. This is a wise methodological tack, since it enables the historian to avoid reductive anachronism while channeling his historical empathy toward an understanding of the past in its own right. It is also wise to take the written documents of the past to mean precisely what they say, short of clear-cut proof to the contrary. There is no historical advantage whatever growing out of the gratuitous assumption that the men of old played tricks on us by systematically hiding their line of thought.

I shall not presume to define here what mathematics is, as that is best left to mathematicians. Besides, there are plenty of definitions available, running the gamut from Bertrand Russell's to Nicolas Bourbaki's.[32] Every reader can easily take his

[30]See G. R. Elton, *The Practice of History* (Sidney: Sidney University Press, 1967), pp. 8–12.

[31]". . . there is more to history than the study of persistent structures and the slow progress of evolution" (Fernand Braudel, *The Mediterranean and the Mediterranean World in the Age of Philip II,* 2 vols., New York: Harper & Row, 1975, Vol. II, p. 901).

[32]". . . mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true" (Bertrand Russell, *Mysticism and Logic and Other Essays,* New York:

pick. But I can say safely what mathematics is not. It is certainly not history. The domain of mathematics is not the idiosyncratic, but, in a very real sense, the nomothetic, since what mathematicians do is to show that from certain assumptions about as yet unidentified objects some conclusions about the same objects will follow necessarily, by rule.

The history of mathematics is history not mathematics. It is the study of the idiosyncratic aspects of the activity of mathematicians who themselves are engaged in the study of the nomothetic, that is, of what is the case by law. If one is to write the history of mathematics, and not the mathematics of history, the writer must be careful not to substitute the nomothetic for the idiosyncratic, that is, not to deal with past mathematics as if mathematics had no past beyond trivial differences in the outward appearance of what is basically an unchangeable hard-core content.

In mathematics (like in anything else) form and content are not independent variables. On the contrary, they mutually condition one another and neither is immune to change. A certain form permits only a certain content, and a new content requires a new form. This is why the methodological approach which casts indiscriminately the algebraic shadow over the garden of Greek mathematics obscures precisely those features which make it *Greek* mathematics. Instead of showing the degree to which it was unlike modern, post-Renaissance mathematics, that approach, by greatly overemphasizing the similarities, prevents an understanding of Greek mathematics in its own right. It also leads in the long run to the untenable view that the Greek mathematicians did not mean what they said, but that they hid "admirably"[33] their line of thought. Coupled with this is the great danger of easily "discerning" problematic or nonexistent influences between mathematical cultures a world apart, simply because when submitted to the algebraic cure all mathematical cultures look alike.

Entrenched as it is, the traditional interpretation of the history of ancient mathematics must give way to a new, more sympathetic, and historically responsive interpretation, simply because the old interpretation has outlived its usefulness and is now an obstacle on the road to a sensitive historical understanding of ancient mathematical texts. After all, like scientific theories, historical theories are tentative attempts to make sense of the past; they are provisional by their very nature, and consequently their authors should not be dreaming hopelessly of endowing them, in God-like fashion, with eternal life and immaculate beatitude.

Otto Neugebauer is right. Speaking of the fact that it was the Hindus and not the Babylonians who introduced a sign for zero to be used *always* whenever required in the writing of numbers, Neugebauer makes the following pertinent remark:

> It seems to me . . . that the awareness of the arbitrariness and the purely conventional, symbolic character of all means of expression does not arise in the framework of a continuous historical development, which, of course, rests on the direct tradition from generation to generation; the awareness of all these things becomes absolutized into fixed and rigid forms which cannot be substantially changed of one's own accord, as this largely transcends the analytical capacity of mankind. Only men who are the heirs of an alto-

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Barnes and Noble, 1971, pp. 59–60) and "A *mathematical theory* . . . contains rules which allow us to assert that certain assemblies of signs are *terms* or relations of the theory, and other rules which allow us to assert that certain assemblies are *theorems* of the theory" (Nicolas Bourbaki, *Elements of Mathematics: Theory of Sets,* Paris/London: Addison-Wesley, 1968, p. 16).

[33]B. L. van der Waerden, *Science Awakening* (Groningen: P. Noordhoff, 1954), p. 172.

gether different historical tradition are able to use freely the foreign means of expression and to recognize both their limits and their potentialities.[34]

Now this seems indeed to be the case with respect to the introduction of the algebraic approach by Viète, Fermat, and Descartes, men of genius belonging to another culture than the Greek, but who managed somehow to discern in what the Greeks had done (geometry) precisely what the Greeks themselves never dreamt about when they were doing it, namely a hidden algebraic structure, which the moderns set about to extract from the Greek texts. This is the true historical origin of the concept "geometrical algebra." It is the intellectual product of foreigners, barbarians, reading Greek mathematical texts in light of their own idiosyncrasies, their own barbarian approach, and "seeing" in it what the Greeks, the autochthons, never put into it, namely, an algebraic train of thought. *Mutatis mutandis,* like the Babylonians with the general concept of zero, the Greeks never came up with a symbolic approach; it remained for the sixteenth- and seventeenth-century Europeans (playing somewhat the role of the Hindus in our comparison), the heirs and at the same time the usurpers of the Greeks, to invent the general symbolic approach and thereby to "perceive" its roots within the confines of Greek geometry.

Though what one calls a thing is, to begin with, merely a convention, once the calling (naming) has been socially accepted, departures from the standard usage without further ado are misleading and can be dangerous. Whatever *algebra* might "really" be, the term as standardly used means something definite, as do most of the words used in common parlance. This is what makes communication possible. A "table" is a table. A "chair" is a chair. Even Freudenthal agrees with that, since he says: "'algebra' has a meaning in everyday language just as 'chair' and 'table' have."[35] Calling, then, a tree "table" is misleading, in spite of the fact that trees can (and quite often do) become tables. (As a matter of fact—and this is crucial—quite often they do not.) By the same token, calling a tree "chair" is misleading. In such an arbitrary naming procedure, one substitutes one of the many potentialities of the object for its reality. This is dangerous, since trees are potentially not just tables or chairs, but also coffins or houses. Calling a tree "table," then, is misleading not only because it takes the potential for the real but also because it neglects all but one of the various potentialities of the object.

Precisely as it is only hindsight that enables one to call legitimately a certain tree "table," or "chair," or "coffin," it is only unwarranted historical hindsight that has enabled scholars to call Greek geometry "algebra," by setting up just one of the potentialities of Greek geometry into a chosen entelechy. There may exist, by divine decree, a chosen people. However, "chosen" entelechies, in the perfectly natural case of multiple potentialities, are *post factum* creations of the mind of the historian-philosopher running rampant, since the whole historical point consists exactly in the necessity to show that in the actual historical process only "the chosen entelechy" has been realized.

[34] *Vorgriechische Mathematik,* p. 78: "Mir scheint . . . dass im Rahmen einer kontinuierlichen geschichtlichen Entwicklung, die ja auf der direkten Tradition von Generation zu Generation beruht, das Bewusstsein der Willkürlichkeit und des rein konventionellen symbolischen Charakters aller Ausdrucksmittel gar nicht ensteht, dass alle diese Dinge zu aboluten und gegebenen Formen werden, die aus freien Stücken wesentlich abzuändern das analytische Vermögen der Menschen weit übersteigt. Erst Menschen die selbst einer ganz anderen geschichtlichen Tradition entstammen, sind imstande, die fremden Ausdrucksmittel frei zu gebrauchen und ihre Schranken wie ihre Möglichkeit zu erkennen."

[35] "What Is Algebra," p. 193.

It is true that names are conventions. But conventions fulfill a very important function, making articulate communication (i.e., intelligent life) possible. Abiding by them enables one to carry on in everyday life. Blatant transgressions against socially accepted conventions, on the other hand, prevent normal communication and can be rather troublesome. It is mere convention to kiss and embrace one's bride at the wedding. Refuse to do it, "because it is a mere convention," and you are in for some real trouble.

The name "algebra," like all names, is a convention (although it has some very definite historical roots). But it means something recognizable in common parlance. Apply it indiscriminately to what is obviously geometry and you have not merely breached a useful convention, you have thereby created a new one, less definite, sharp, and useful than the one you violated, since it substitutes potentiality for reality. And although this is possible, it is wrong historically, since history deals with reality (what happened) and not with potentiality (what could have happened logically). The approach of Freudenthal, van der Waerden, and their cohort substitutes logic for history.[36]

[36] Here, I have in mind Andre Weil's unprecedented missive to the editor of the *Archive for History of Exact Sciences,* entirely repetitive in its few *non ad hominem* passages of the arguments of van der Waerden and Freudenthal: "Who Betrayed Euclid?" *Arch. Hist. Exact Sci.,* 1978, *19*:91–93. Concerning this letter, the less said the better. In adopting this position, I am guided by Simone Weil's words in her sensitive and penetrating essay on the *Iliad (The Iliad or the Poem of Force,* Wallingford, Pa.: Pendle Hill, n.d., pp. 3, 36): "To define force—it is that *x* that turns anybody who is subjected to it into a *thing.* Exercised to the limit, it turns man into a thing in the most literal sense: it makes a corpse out of him. Somebody was here, and the next minute there is nobody here at all;" And: "The man who does not wear the armor of the lie cannot experience force without being touched by it to the very soul. Grace can prevent this touch from corrupting him, but it cannot spare him the wound."

METHODOLOGY, PHILOLOGY, AND PHILOSOPHY

By Wilbur Knorr*

IN A NOTE to his article "The Philosophical Sense of Theaetetus' Mathematics" (*Isis,* 1978, *69*:489–513, n. 88), M. F. Burnyeat raises some ostensibly damaging observations against the argument in my book *The Evolution of the Euclidean Elements* (Dordrecht: Reidel, 1975). I would like to make a few remarks in the hope of correcting at once a number of basic misconceptions.

A major issue seems to center on the interpretation of a single sentence in Plato's *Theaetetus*: is it to be rendered "in this case [i.e., 17] he [Theodorus] for some reason *came to a standstill* [*enescheto*]," as most commentators have chosen; or, alternatively, "in this case for some reason he *encountered difficulty,"* as I have taken it, supported by the textual analysis given by R. Hackforth (*Mnemosyne,* 1957, *10*:128;

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The concept distinguishes these forms of contention from the everyday acts of resistance explored by James C. Scott, interstate warfare, and forms of contention employed entirely within institutional settings, such as elections or sports. Historical sociologist Charles Tilly defines contentious politics as "interactions in which actors make claims bearing on someone else's interest, in which governments appear either as targets, initiators of claims, or third parties."[1]. "Development is contentious, and contentions over theories and practices of development are unlikely to end soon. Peet and Hartwick do not mince words as they offer a provocative critique of conventional, poststructuralist, and postdevelopmentalist theories. Their critical modernist perspective refuses to abandon hope for a better society through truly democratic development.