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On multiplicatively closed subsets of normed algebras ☆

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A B S T R A C T

It is well known that, if S is a bounded and multiplicatively closed subset of an associative normed algebra $(A, \|\cdot\|)$, then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||s||| \leqslant 1$ for every $s \in S$. Although associativity is not an essential requirement in this result, it is easy to find examples of nonassociative normed algebras A where such a result fails. Actually, it can fail even if the subset S is reduced to a nonzero idempotent. We prove that it remain true in the nonassociative setting whenever the subset S is assumed to be contained in the nucleus of A. In the particular case that the subset S reduces to a nonzero nuclear idempotent p, we show that the equivalent algebra norm $|||\cdot|||$ above can be chosen so that p becomes a strongly exposed point of the closed unit ball of $(A, |||\cdot|||)$. We study those (possibly nonassociative) normed algebras A satisfying the "norm-one boundedness property" (in short, NBP), which means that, as happened in the associative case, for every bounded and multiplicatively closed subset S of A, there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||s||| \leqslant 1$ for every $s \in S$. We show that absolute-valued algebras, JB-algebras, and nilpotent normed algebras fulfil the NBP. We also show that, if an anti-commutative complete normed algebraic algebra A satisfies the NBP, then there exists $n \in \mathbb{N}$ such that $L_a^n = 0$ for every $a \in A$, where L_a denotes the operator of left multiplication by a. It follows from a celebrated theorem of E.I. Zel'manov on the so-called Engel Lie algebras that a complete normed algebraic Lie algebra satisfies the NBP if and only if it is nilpotent.

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1. Introduction

By an algebra norm on a (possibly nonassociative) real or complex algebra A we mean a norm $\|\cdot\|$ on (the vector space of) A satisfying $\|ab\| \leqslant \|a\|\|b\|$ for all $a, b \in A$. By a normed algebra we mean a real or complex algebra endowed with an algebra norm. A well-known result in the theory of associative normed algebras is the following theorem (see [7, Theorem I.4.1]).

Theorem 1.1. *Let A be an associative normed algebra, and let S be a bounded and multiplicatively closed subset of A. Then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||s||| \leqslant 1$ for every $s \in S$.*

As an immediate consequence, we have the following.

Corollary 1.2. *Let A be an associative normed algebra, and let p be a nonzero idempotent in A. Then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||p||| = 1$.*

It is easily realized that neither Theorem 1.1 nor even Corollary 1.2 remain true if the assumption of associativity is removed (see for instance Example 2.1). Thus, the aim of this paper is to discuss the validity of Theorem 1.1 in the nonassociative setting. As we will explain in some detail in what follows, this will depend on the goodness of the bounded and multiplicatively closed subset S and/or the goodness of the (possibly nonassociative) normed algebra A (which, in its turn, could depend on either the purely algebraic structure of A or the behaviour of the norm). On the other hand, the methods of proof in the nonassociative discussion of Theorem 1.1 had to be different from those applied in the associative setting, so that we obtain some algebra renorming results which seem to be new even in the associative case.

Our first main result states that Theorem 1.1 remains true if associativity of A is altogether removed, but the bounded and multiplicatively closed subset S is assumed to be contained in the nucleus of A (Theorem 2.3). We recall that the nucleus of an algebra A is defined as the set of those elements of A which associate with any two elements of A. Thus any subset of an associative algebra is contained in the nucleus, and hence Theorem 2.3 generalizes Theorem 1.1. On the other hand, every nuclear element a of an algebra A generates an associative subalgebra, so that, in the case that A is normed, we can consider the spectral radius of a, defined as usual by $r(a) := \inf\{\|a^n\|^{\frac{1}{n}}:\ n \in \mathbb{N}\}$. As a consequence of Theorem 2.3, we prove that, if A is a normed algebra, and if a is any nuclear element of A, then $r(a)$ coincides with the infimum of the set of values at a of all equivalent algebra norms on A (Corollary 2.10). This becomes a nonassociative generalization of [7, Corollary I.4.2].

Different classes of nonassociative algebras which are "close" to the associative ones have appeared in the literature. The most relevant ones are those considered in the following chain of implications:

$$\text{Associative} \Longrightarrow \text{Alternative} \Longrightarrow \text{Generalized Standard}$$
$$\Longrightarrow \text{Noncommutative Jordan} \Longrightarrow \text{Power-associative}.$$

The reader is referred to [29] for the definition of generalized standard algebras, and to Schafer's book [28] for the definition of the remaining classes of algebras. We prove that Corollary 1.2 remains true if the assumption that A is associative is relaxed to the one that A is generalized standard (Theorem 3.2), but not to the one that A is noncommutative Jordan (Example 3.5). Unfortunately, we do not know whether Theorem 1.1 remains true if the assumption that A is associative is relaxed to the one that A is generalized standard, nor even to the one that A is alternative (see 6.1 for more details).

Among the properties of a point u of the unit sphere of a normed space (relative to its closed unit ball), we collect here those considered in the following chain of implications:

$$u \text{ is a strongly exposed point} \Longrightarrow u \text{ is a denting point}$$
$$\Longrightarrow u \text{ is a strongly extreme point}.$$

Precise definitions of these properties, as well as references and comments concerning their relation with normed algebras, can be found in Remark 2.9 and the comment immediately before Lemma 2.7. It is known that if A is a "norm-unital" normed algebra (i.e., a normed algebra with a unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$), then $\mathbf{1}$ is a strongly extreme point (of the closed unit ball of A), but need not be a denting point (much less a strongly exposed point). Indeed, the norm-unital Banach algebra $\mathcal{L}(H)$ of all bounded linear operators on any infinite-dimensional Hilbert space H has no denting point. We prove in Theorem 2.8 that, if A is a normed algebra with a unit $\mathbf{1}$, then there exists an equivalent algebra norm $|||\cdot|||$ (which is a dual norm whenever A is a dual Banach space) such that $(A, |||\cdot|||)$ is norm-unital, and $\mathbf{1}$ becomes a strongly exposed point of $B_{(A,|||\cdot|||)}$. Moreover, if A is in fact norm-unital, then the norm $|||\cdot|||$ above can be chosen arbitrarily close to the original norm. We also prove that, if A is a normed algebra, if p is a nonzero idempotent in A, and if either A is standard generalized or p is nuclear (both requirements being automatically fulfilled whenever A is associative), then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||p||| = 1$, and p becomes a strongly exposed point of $B_{(A,|||\cdot|||)}$ (Corollary 3.4 and Proposition 4.2). All algebra renorming results reviewed in the present paragraph seem to be new even in the associative setting.

In Section 5 we realize that associativity is not an essential requirement in Theorem 1.1, and hence we introduce and study the class of those normed algebras A satisfying the "norm-one boundedness property" (in short, NBP), which means that for every bounded and multiplicatively closed subset S of A, there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||s||| \leqslant 1$ for every $s \in S$. We prove in Proposition 5.1 that (a strong form of) the NBP is fulfilled by those normed algebras A satisfying $\|a^2\| = \|a\|^2$ for every $a \in A$, and note that among such normed algebras we find all absolute-valued algebras [27], and all *JB*-algebras [13], so that the NBP is certainly fulfilled by "many" nonassociative normed algebras. We define the spectral radius $r(a)$ of an arbitrary element a of a normed algebra A, note that equivalent algebra norms on A give the same spectral radius for a, and prove that, if A satisfies the NBP, then $r(a)$ coincides with the infimum of the set of values at a of all equivalent algebra norms on A (Proposition 5.4). As a consequence, we show that the so-called "nearly absolute-valued algebras" [15] need not satisfy the NBP (Proposition 5.6).

We prove that all nilpotent normed algebras satisfy the NBP (Proposition 5.8), and that, if an anti-commutative complete normed algebraic algebra A satisfies the NBP, then there exists $n \in \mathbb{N}$ such that $L_a^n = 0$ for every $a \in A$, where L_a denotes the operator of left multiplication by a (Proposition 5.10). It follows from a celebrated theorem of E.I. Zel'manov [30] that a complete normed algebraic Lie algebra satisfies the NBP if and only if it is nilpotent (Theorem 5.11). As a consequence, a finite-dimensional normed Lie algebra satisfies the NBP if and only if it is nilpotent (Corollary 5.12).

2. General nonassociative algebras

As the next easy example shows, neither Theorem 1.1 nor even its consequence (Corollary 1.2) remain true if associativity is removed.

Example 2.1. Let λ be a real number with $\lambda > 1$. Then there exists a two-dimensional commutative normed algebra A with an idempotent p satisfying $\|p\| = \lambda$ and $|||p||| \geqslant \lambda$ for every algebra norm $|||\cdot|||$ on A. Indeed, take a vector space with basis $\{p, q\}$, convert it into an algebra with multiplication table

$$\begin{array}{c|cc} & p & q \\ \hline p & p & \lambda q \\ q & \lambda q & 0 \end{array} \tag{2.1}$$

and define a norm on it by $\|\alpha p + \beta q\| := \lambda|\alpha| + |\beta|$. It is easily realized that $\|\cdot\|$ becomes an algebra norm, giving rise in this way to a two-dimensional commutative normed algebra A with an idempotent p satisfying $\|p\| = \lambda$. Moreover, for every algebra norm $|||\cdot|||$ on A we have $\lambda|||q||| = |||pq||| \leqslant |||p||| |||q|||$, and hence $\lambda \leqslant |||p|||$.

The next lemma will be useful to get reasonable nonassociative generalizations of Theorem 1.1. For any normed space X, we denote by B_X the closed unit ball of X.

Lemma 2.2. *Let A be a normed algebra, and let S be a subset of A. Then there exists an equivalent algebra norm $|||\cdot|||$ on A satisfying $|||s|||\leqslant 1$ for every $s\in S$ if (and only if) there exists $\varepsilon>0$ such that the multiplicatively closed subset of A generated by $(\varepsilon B_A)\cup S$ is bounded.*

Proof. Assume that there exists $\varepsilon>0$ such that the multiplicatively closed subset of A generated by $(\varepsilon B_A)\cup S$ (say T) is bounded (by $M>0$, say). Since the absolutely convex hull of T (say U) is multiplicatively closed, bounded (by $M>0$), and absorbent (because in fact it contains εB_A), we can argue as in [7, Proposition 1.9] to obtain that the Minkowski functional of U (say $|||\cdot|||$) is an algebra norm on A satisfying

$$\frac{1}{M}\|\cdot\|\leqslant|||\cdot|||\leqslant\frac{1}{\varepsilon}\|\cdot\|$$

and $|||s|||\leqslant 1$ for every $s\in S$. $\square$

Let A be an algebra. For $a,b,c\in A$, we put

$$[a,b,c]:=(ab)c-a(bc).$$

The nucleus of A is defined as the set of those elements $a\in A$ such that $[a,A,A]=[A,a,A]=[A,A,a]=0$. A reasonable nonassociative generalization of Theorem 1.1 is the following.

Theorem 2.3. *Let A be a normed algebra, and let S be a bounded and multiplicatively closed subset of A contained in the nucleus of A. Then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||s|||\leqslant 1$ for every $s\in S$.*

Proof. Let $M\geqslant 1$ be a bound for S. We put $\varepsilon:=\frac{1}{M}$, and claim that the family $\mathcal{F}$ of subsets of A given by

$$\mathcal{F}:=\{S,\varepsilon B_A,\varepsilon B_A S,\varepsilon S B_A,\varepsilon S B_A S\}$$

(where $SB_AS:=(SB_A)S=S(B_AS)$ by nuclearity of S) has the property that, whenever X and Y are in $\mathcal{F}$, we have $XY\subseteq Z$ for some $Z\in\mathcal{F}$. Indeed, $\mathcal{F}$ fulfils such a property according to the following table:

$X\diagdown Y$	S	εB_A	$\varepsilon B_A S$	$\varepsilon S B_A$	$\varepsilon S B_A S$
S	S	$\varepsilon S B_A$	$\varepsilon S B_A S$	$\varepsilon S B_A$	$\varepsilon S B_A S$
εB_A	$\varepsilon B_A S$	εB_A	εB_A	εB_A	$\varepsilon B_A S$
$\varepsilon B_A S$	$\varepsilon B_A S$	εB_A	$\varepsilon B_A S$	εB_A	$\varepsilon B_A S$
$\varepsilon S B_A$	$\varepsilon S B_A S$	εB_A	$\varepsilon S B_A$	$\varepsilon S B_A$	$\varepsilon S B_A S$
$\varepsilon S B_A S$	$\varepsilon S B_A S$	$\varepsilon S B_A$	$\varepsilon S B_A S$	$\varepsilon S B_A$	$\varepsilon S B_A S$

(2.2)

The above table can be easily verified by applying that S is nuclear and multiplicatively closed, and that $\varepsilon=\frac{1}{M}\leqslant 1$ (which implies $\varepsilon S\subseteq B_A$). As a sample, we show that the inclusion $XY\subseteq Z$ holds in the case that $X=Y=Z=\varepsilon SB_AS$. Indeed, we have

$$\begin{aligned}(\varepsilon SB_AS)(\varepsilon SB_AS)&=\big[(\varepsilon SB_AS)(\varepsilon S)\big](B_AS)=\big[(\varepsilon SB_A)(\varepsilon SS)\big](B_AS)\\&\subseteq\big[(\varepsilon SB_A)(\varepsilon S)\big](B_AS)\subseteq\big[(\varepsilon SB_A)B_A\big](B_AS)\\&=\big[\varepsilon S(B_AB_A)\big](B_AS)\subseteq(\varepsilon SB_A)(B_AS)=\varepsilon S(B_AB_A)S\subseteq\varepsilon SB_AS.\end{aligned}$$

It follows from the definition of $\mathcal{F}$ and the claim just proved that

$$T := S \cup (\varepsilon B_A) \cup (\varepsilon B_A S) \cup (\varepsilon S B_A) \cup (\varepsilon S B_A S)$$

coincides with the multiplicatively closed subset of A generated by $(\varepsilon B_A) \cup S$. Since T is bounded (by M), Lemma 2.2 applies. □

It follows from Theorem 2.3 that, if p is a nonzero nuclear idempotent in a normed algebra A, then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||p||| = 1$ (a nonassociative generalization of Corollary 1.2). In particular, we have the following result, first proved by F.G. Ocaña [22].

Corollary 2.4. *Let A be a nonzero normed algebra with a unit $\mathbf{1}$. Then there exists an equivalent algebra norm $|||\cdot|||$ on A satisfying $|||\mathbf{1}||| = 1$.*

Ocaña's norm $|||\cdot|||$ in Corollary 2.4 can be explicitly given. Indeed, the set $T := B_A \cup \{\mathbf{1}\}$ is multiplicatively closed, so that, according to the proof of Lemma 2.2, we can take $|||\cdot|||$ equal to the Minkowski functional of the absolutely convex hull of T, and then we have

$$|||a||| = \inf\{|\lambda| + \|a - \lambda\mathbf{1}\|: \lambda \in \mathbb{K}\} \tag{2.3}$$

for every $a \in A$. Here $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$, depending on whether A is real or complex.

Remark 2.5. Let A be a normed algebra, let S be a bounded and multiplicatively closed subset of A contained in the nucleus of A, and put $M := \max\{1, \sup\{\|s\|: s \in S\}\}$. Looking at the proofs of Lemma 2.2 and Theorem 2.3, we realize that the algebra norm $|||\cdot|||$ on A given by Theorem 2.3 satisfies $\frac{1}{M}\|\cdot\| \leqslant |||\cdot||| \leqslant M\|\cdot\|$.

For a normed space X, we denote by S_X the unit sphere of X, and by X^* the (topological) dual of X. By $\Re(z)$ (respectively, $\Im(z)$) we mean the real (respectively, imaginary) part of the (eventually real) complex number z.

Lemma 2.6. *Let X be a normed space, and let u and f be in X and X^*, respectively, with $\|u\| < 1 = f(u)$. For x in X, put*

$$|||x||| := \max\{\|x\|, |f(x)|\}.$$

Then $|||\cdot|||$ becomes an equivalent norm on X satisfying $|||u||| = 1$. Moreover, $|||\cdot|||$ is Fréchet differentiable at u with Fréchet derivative equal to $\Re \circ f$.

Proof. The first assertion in the conclusion is clear. To prove the second assertion, note that, since $\|u\| < |f(u)|$, we have $\|u + h\| < |f(u + h)|$ for $h \in X$ with $|||h|||$ small enough. It follows that

$$\begin{aligned}\lim_{h\to 0} \frac{|||u+h||| - 1 - \Re(f(h))}{|||h|||} &= \lim_{h\to 0} \frac{|f(u+h)| - 1 - \Re(f(h))}{|||h|||} \\ &= \lim_{h\to 0} \frac{[\Im(f(h))]^2}{[|f(u+h)| + 1 + \Re(f(h))]|||h|||} = \frac{1}{2}\lim_{h\to 0} \frac{[\Im(f(h))]^2}{|||h|||} = 0.\end{aligned}$$

Therefore the mapping $\Re \circ f$ is the Fréchet derivative of $|||\cdot|||$ at u. □

Let X be a normed space. An element $v \in B_X$ is said to be a strongly exposed point (of B_X) if there exists $g \in S_{X^*}$ with the property that, whenever (x_n) is a sequence in B_X such that $(g(x_n)) \to 1$, we have $(x_n) \to v$. When the functional g must be emphasized, we say that v is strongly exposed by g. In the case that X is a dual Banach space, we say that the element $v \in B_X$ is a w^*-strongly exposed point of B_X if the functional g above can be chosen w^*-continuous. Now, let $\mathcal{N}(X)$ denote the set of all equivalent norms on X. We recall that $\mathcal{N}(X)$ becomes naturally a metric space under the distance $\Delta(\|.\|_1, \|.\|_2) := \log(k)$, where k is the smallest positive number satisfying $k^{-1}\|.\|_1 \leqslant \|.\|_2 \leqslant k\|.\|_1$.

Lemma 2.7. *Let Y be a normed space, let v be in Y with $\|v\| > 1$, and let $|||\cdot|||$ stand for the Minkowski functional on Y of the absolutely convex hull of $B_Y \cup \{v\}$. Then we have*:

(1) *$|||\cdot|||$ is an equivalent norm on Y satisfying $|||v||| = 1$ and*

$$\Delta\big(|||\cdot|||, \|\cdot\|\big) = \log\|v\|. \tag{2.4}$$

(2) *v is a strongly exposed point of $B_{(Y,|||\cdot|||)}$.*
(3) *If Y is a dual Banach space (completeness of the predual is not required), then $|||\cdot|||$ is a dual norm, and v is a w^*-strongly exposed point of $B_{(Y,|||\cdot|||)}$.*

Proof. Assertion (1) is straightforward.

To prove (2), let T stand for the absolutely convex hull of $B_Y \cup \{v\}$ in Y, and note that, as a consequence of the definition of $|||\cdot|||$, we have

$$\big\{y \in Y\colon\ |||y||| < 1\big\} \subseteq T \subseteq B_{(Y,|||\cdot|||)}. \tag{2.5}$$

This implies that, for every $x \in X := Y^*$, we have

$$|||x||| = \sup\big\{\big|x(y)\big|\colon\ y \in T\big\} = \sup\big\{\big|x(y)\big|\colon\ y \in B_Y \cup \{v\}\big\} = \max\big\{\|x\|, \big|x(v)\big|\big\}.$$

Now, put $f := v \in Y \subseteq Y^{**} = X^*$, and note that, since $\|f\| > 1$, there exists $u \in X$ with $\|u\| < 1 = f(u)$ (indeed, the range of the open unit ball of X under f is the open unit ball in the base field with center 0 and radius $\|f\|$). It follows from Lemma 2.6 that $|||\cdot|||$ is Fréchet differentiable at u with Fréchet derivative equal to $\Re(f)$. This implies that $f = v$ is strongly exposed by u in $(Y, |||\cdot|||)$ (see for example [10, Lemma 8.4]).

To conclude the proof, assume that Y is a dual Banach space. Since T is the absolutely convex hull of $B_Y \cup \{v\}$, it is also the convex hull of $B_Y \cup (B_{\mathbb{K}} v)$, where $\mathbb{K}$ stands for the base field for A (equal to $\mathbb{R}$ or $\mathbb{C}$). On the other hand, both B_Y and $B_{\mathbb{K}} v$ are w^*-compact convex subsets of Y (the first, by Alaoglu's theorem). It follows that T is w^*-compact, and hence norm-closed in Y. By applying (2.5), we deduce that $B_{(Y,|||\cdot|||)} = T$, and hence that $B_{(Y,|||\cdot|||)}$ is w^*-closed in Y. As it is well known, this is equivalent to say that $|||\cdot|||$ is a dual norm on Y. To prove that v is a w^*-strongly exposed point of $B_{(Y,|||\cdot|||)}$ it is enough to show that the element $u \in X = Y^*$ in the above paragraph can be taken into the predual (say Y_*) of Y. But this follows because $\|v\| > 1$ and the range of the open unit ball of Y_* under v is the open unit ball in the base field with center 0 and radius $\|f\|$. □

Theorem 2.8. *Let $(A, \|\cdot\|)$ be a nonzero normed algebra with a unit $\mathbf{1}$. Then there exists an equivalent algebra norm $|||\cdot|||$ on A satisfying that $|||\mathbf{1}||| = 1$ and that $\mathbf{1}$ is a strongly exposed point of $B_{(A,|||\cdot|||)}$. Moreover we have*:

(1) *If $\|\mathbf{1}\| = 1$, then the norm $|||\cdot|||$ above can be chosen in such a way that $\Delta(|||\cdot|||, \|\cdot\|)$ is arbitrarily small.*
(2) *If A is a dual Banach space, then the norm $|||\cdot|||$ above can be chosen among dual norms on A, and such that $\mathbf{1}$ becomes in fact a w^*-strongly exposed point of $B_{(A,|||\cdot|||)}$.*
(3) *If $\|\mathbf{1}\| = 1$, and if A is a dual Banach space, then the norm $|||\cdot|||$ above can be chosen among dual norms on A, and in such a way that $\Delta(|||\cdot|||, \|\cdot\|)$ is arbitrarily small, and that $\mathbf{1}$ becomes in fact a w^*-strongly exposed point of $B_{(A,|||\cdot|||)}$.*

Proof. First assume that $\|\mathbf{1}\| > 1$. Let $|||\cdot|||$ be the Minkowski functional on A of the absolutely convex hull of $B_A \cup \{\mathbf{1}\}$. We know that $|||\cdot|||$ is an equivalent algebra norm on A satisfying $|||\mathbf{1}||| = 1$. On the other hand, by Lemma 2.7(2), $\mathbf{1}$ becomes a strongly exposed point of $B_{(A,|||\cdot|||)}$.

Now assume that $\|\mathbf{1}\| = 1$. Let k be any real number with $k > 1$, and put $\|\cdot\|_k := k\|\cdot\|$, so that $\|\cdot\|_k$ becomes an equivalent algebra norm on A such that $\|\mathbf{1}\|_k = k > 1$. By the preceding paragraph, there exists an equivalent algebra norm $|||\cdot|||_k$ on A satisfying that $|||\mathbf{1}|||_k = 1$ and that $\mathbf{1}$ is a strongly exposed point of $B_{(A,|||\cdot|||_k)}$. Moreover, by (2.4), we have

$$\Delta\big(|||\cdot|||_k, \|\cdot\|_k\big) = \log\|\mathbf{1}\|_k = \log k.$$

Since also $\Delta(\|\cdot\|_k, \|\cdot\|) = \log k$, we deduce

$$\Delta\big(|||\cdot|||_k, \|\cdot\|\big) \leqslant \Delta\big(|||\cdot|||_k, \|\cdot\|_k\big) + \Delta\big(\|\cdot\|_k, \|\cdot\|\big) \leqslant 2\log k.$$

It follows that, for k close enough to 1, $\Delta(|||\cdot|||_k, \|\cdot\|)$ is arbitrarily small.

Now assume that $\|\mathbf{1}\| > 1$ and that A is a dual Banach space. Let $|||\cdot|||$ be as in the first paragraph of the proof, so that $|||\cdot|||$ becomes an equivalent algebra norm on A satisfying that $|||\mathbf{1}||| = 1$. On the other hand, by Lemma 2.7(3), $|||\cdot|||$ is a dual norm, and $\mathbf{1}$ is a w^*-strongly exposed point of $B_{(A,|||\cdot|||)}$.

Finally assume that $\|\mathbf{1}\| = 1$ and that A is a dual Banach space. Let k be any real number with $k > 1$, and let $\|\cdot\|_k$ and $|||\cdot|||_k$ be as in the second paragraph of the proof, so that both $\|\cdot\|_k$ and $|||\cdot|||_k$ are equivalent algebra norms on A, and satisfy $\|\mathbf{1}\|_k = k > 1$ and $|||\mathbf{1}|||_k = 1$. Moreover, for k close enough to 1, $\Delta(|||\cdot|||_k, \|\cdot\|)$ is arbitrarily small. Since $|||\cdot|||_k$ is the corresponding norm to $\|\cdot\|_k$ in the first paragraph of the proof (when $|||\cdot|||_k$ and $\|\cdot\|_k$ replace $|||\cdot|||$ and $\|\cdot\|$, respectively), and $\|\cdot\|_k$ is clearly a dual norm, we can apply the third paragraph to obtain that $|||\cdot|||_k$ is a dual norm, and that $\mathbf{1}$ becomes actually a w^*-strongly exposed point of $B_{(A,|||\cdot|||_k)}$. □

Remark 2.9. Let X be a normed space, and let u be in S_X. It is well known that u is strongly exposed by $g \in S_{X^*}$ if and only if $g(u) = 1$ and, for $0 < \delta < 1$, the diameter of the "slice"

$$S(X, g, \delta) := \big\{x \in B_X \colon \Re\big(g(x)\big) > 1 - \delta\big\}$$

tends to 0 as $\delta \to 0$. Therefore, if u is a strongly exposed point, then u is a denting point (of B_X), which means that there are slices of arbitrarily small diameter which contain u. On the other hand, if u is a denting point, then u is a strongly extreme point (of B_X), which means that, whenever (x_n) and (y_n) are sequences in B_X such that $(\frac{x_n+y_n}{2}) \to u$, we have $(x_n) \to u$ and $(y_n) \to u$ (see [18, p. 169]).

Now, let A be a normed algebra with a unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$. Then $\mathbf{1}$ is a strongly extreme point (of B_A). If A is associative, this follows from [6, Theorem 4.5] after passing to completion and complexification if necessary. The result remains true even if A is not associative. Indeed, the mapping $a \to L_a$ (where L_a denotes the operator of left multiplication by a) becomes a unit-preserving linear isometry from A to the associative normed algebra of all bounded linear operators on A. However, even if A is associative, $\mathbf{1}$ need not be a denting point (of B_A). Indeed, the Banach algebra $\mathcal{L}(H)$ of all bounded linear operators on any infinite-dimensional Hilbert space H has no denting point [11]. Actually, all slices (and, more generally, all nonempty relatively weakly open subsets) of the closed unit ball of $\mathcal{L}(H)$ have diameter equal to 2 (see [5] and [4]). The reader is referred to [14] for quantitative versions of the fact that the units of norm-unital normed algebras are strongly extreme points, and to [3,6,20,23,24] for other interesting geometrical properties of the units of norm-unital normed algebras.

Let A be a normed algebra, let a be a nuclear element of A such that $a^2 = 0$, and let $\varepsilon > 0$. Then there exists an equivalent algebra norm $|||\cdot|||$ on A satisfying $|||a||| \leqslant \varepsilon$. Indeed, the set $\{0, \frac{a}{\varepsilon}\}$ is bounded and multiplicatively closed, and is contained in the nucleus of A, and therefore Theorem 2.3 applies. In what follows, we are going to generalize the fact just reviewed.

Let A be a normed algebra, and let a be in A such that the subalgebra of A generated by a is associative. The spectral radius of a, denoted by $r(a)$, is defined by

$$r(a) := \inf\left\{ \left\| a^n \right\|^{\frac{1}{n}} \colon n \in \mathbb{N} \right\}.$$

It is well known that

$$r(a) = \lim_{n \to \infty} \left\{ \left\| a^n \right\|^{\frac{1}{n}} \right\}$$

(see for example [7, Proposition I.2.8]), and hence that equivalent algebra norms on A give the same spectral radius for a. As a consequence, we have $r(a) \leqslant |||a|||$ for every equivalent algebra norm $|||\cdot|||$ on A. Now, let $\mathrm{En}(A)$ (respectively, $\mathrm{Eun}(A)$) stand for the set of all equivalent algebra norms on A (respectively, the set of all equivalent algebra norms $|||\cdot|||$ on A such that $|||\mathbf{1}||| = 1$ when A has a unit $\mathbf{1}$), and note that, since the nucleus of A is an associative subalgebra of A [28, p. 13], the requirement done on a (that the subalgebra of A generated by a is associative) is fulfilled in particular when a is nuclear. We have the following.

Corollary 2.10. *Let A be a normed algebra, and let a be a nuclear element of A. Then we have*

(1) $r(a) = \inf\{|||a|||\colon |||\cdot||| \in \mathrm{En}(A)\}$.
(2) *If A has a unit* $\mathbf{1}$, *then we have in fact*

$$r(a) = \inf\left\{|||a|||\colon |||\cdot||| \in \mathrm{Eun}(A)\right\}.$$

Proof. To prove (1), it is enough to show that, if $r(a) < 1$, then there exists $|||\cdot||| \in \mathrm{En}(A)$ such that $|||a||| \leqslant 1$. But, if $r(a) < 1$, then we have $\|a^n\| < 1$ for n big enough, and hence the set $S := \{a^n\colon n \in \mathbb{N}\}$ is bounded. Since S is multiplicatively closed, and is contained in the nucleus of A, the proof is concluded by applying Theorem 2.3.

The proof of (2) is similar, by replacing $\mathrm{En}(A)$ with $\mathrm{Eun}(A)$, and noticing that, if $r(a) < 1$, then the set $\{\mathbf{1}\} \cup \{a^n\colon n \in \mathbb{N}\}$ is bounded, multiplicatively closed, and nuclear. □

When the algebra A in the above corollary is associative, the requirement that the element a lies in the nucleus is automatically fulfilled, and hence can be omitted. Thus we obtain the well-known associative forerunner of our result (see [7, Corollary I.4.2]). For another independent nonassociative generalization of [7, Corollary I.4.2], see Proposition 5.4 below. Noticing that nonzero idempotents generate associative subalgebras, and that, in the normed case, they have spectral radius equal to 1, Example 2.1 shows that Corollary 2.10 need not remain true if the requirement that the element a lies in the nucleus is removed.

3. Nearly associative algebras

Different classes of nonassociative algebras which are "close" to the associative ones have appeared in the literature. We summarize in the following diagram the most relevant ones, as well as the relation between them.

$$
\begin{array}{ccccc}
 & & \text{Associative} & & \\
 & \swarrow\!\!\swarrow & & \searrow\!\!\searrow & \\
 & \text{Alternative} & & \text{Standard} & \\
 & \Downarrow & \swarrow\!\!\swarrow & \Downarrow & \\
 & \text{Generalized standard} & & \text{Accessible} & \\
 & \Downarrow & & \Downarrow & \\
 & \text{Noncommutative Jordan} & & \text{Generalized Accessible} & \\
 & \Downarrow & \searrow\!\!\searrow & \Downarrow & \\
 & \text{Power-associative} & & \text{Flexible} &
\end{array}
\tag{3.1}
$$

Alternative algebras are defined as those algebras satisfying the "left alternative law" $a^2b = a(ab)$ and the "right alternative law" $ba^2 = (ba)a$. By Artin's theorem [28, Theorem 3.1], an algebra A is alternative (if and) only if, for all $a, b \in A$, the subalgebra of A generated by $\{a, b\}$ is associative. Flexible algebras are defined as those algebras satisfying the "flexibility" condition $(ab)a = a(ba)$. Following [28, p. 141], we define noncommutative Jordan algebras as those flexible algebras satisfying the "Jordan identity" $(ab)a^2 = a(ba^2)$. Noncommutative Jordan algebras which are commutative are simply called Jordan algebras. Power-associative algebras are defined as those algebras A such that for every $a \in A$, the subalgebra of A generated by a is associative.

Standard (respectively, generalized standard) algebras are defined by a suitable finite set of identities [1] (respectively, [29]), and, roughly speaking, they compose the minimum class of algebras containing all associative (respectively, alternative) algebras and all Jordan algebras. In its turn, accessible (respectively, generalized accessible) algebras are also defined by a suitable finite set of identities [16] (respectively, [17]), with the aim of composing the minimum class of algebras containing all associative (respectively, alternative) algebras and all commutative algebras.

As Example 2.1 shows, neither Theorem 1.1 nor even Corollary 1.2 remain true if associativity is relaxed to accessibility. The main goal of this section is to prove that Corollary 1.2 remains true if the assumption that A is associative is relaxed to the one that A is a generalized standard algebra.

Lemma 3.1. *Let A be a generalized standard algebra, and let p be an idempotent in A. Then A has a Peirce decomposition (vector space direct sum)*

$$A = A_1 \oplus A_{10} \oplus A_{\frac{1}{2}\frac{1}{2}} \oplus A_{01} \oplus A_0, \tag{3.2}$$

where, for $i = 0, 1$, A_i is defined by

$$A_i := \{x \in A\colon\ px = xp = ix\},$$

whereas, for $i, j = 0, \frac{1}{2}, 1$ with $i + j = 1$, A_{ij} is defined by

$$A_{ij} := \{x \in A\colon\ px = ix,\ xp = jx\}.$$

If in addition A is normed, then the direct sum $A = A_1 \oplus A_{10} \oplus A_{\frac{1}{2}\frac{1}{2}} \oplus A_{01} \oplus A_0$ is topological.

Proof. The first conclusion in the lemma is a part of [29, Theorem 5]. To prove the second conclusion, let us recall that, if T a linear operator on A satisfying

$$(T - 1)\left(T - \frac{1}{2}\right)T = 0, \tag{3.3}$$

then the operators

$$P_1(T) := 2\left(T - \frac{1}{2}\right)T,$$

$$P_{\frac{1}{2}}(T) := -4(T-1)T,$$

$$P_0(T) := 2(T-1)\left(T - \frac{1}{2}\right)$$

are pairwise orthogonal linear projections on A such that

$$P_1(T) + P_{\frac{1}{2}}(T) + P_0(T) = 1.$$

On the other hand, it follows from the proof of [29, Theorem 5], the link in [29] to [21], and the link in [21] to [1], that condition (3.3) is fulfilled for T equal to L_p, R_p, or $M_p := \frac{1}{2}(L_p + R_p)$ (where L_p and R_p denote the operators of left and right, respectively, multiplication by p), and that the projections P_1, P_{10}, $P_{\frac{1}{2}\frac{1}{2}}$, P_{01}, and P_0 from A onto A_1, A_{10}, $A_{\frac{1}{2}\frac{1}{2}}$, A_{01}, and A_0, respectively, corresponding to the decomposition (3.2) are given by

$$\begin{aligned} P_1 &= P_1(M_p),\\ P_{10} &= P_1(L_p)P_0(R_p),\\ P_{\frac{1}{2}\frac{1}{2}} &= P_{\frac{1}{2}}(L_p)P_{\frac{1}{2}}(R_p),\\ P_{01} &= P_0(L_p)P_1(R_p),\\ P_0 &= P_0(M_p). \end{aligned} \tag{3.4}$$

Now, in the case that A is normed, the operators L_p, R_p, and M_p are continuous, so the projections above are continuous, and so the direct sum $A = A_1 \oplus A_{10} \oplus A_{\frac{1}{2}\frac{1}{2}} \oplus A_{01} \oplus A_0$ is topological. □

Theorem 3.2. *Let A be a normed generalized standard algebra, and let p be a nonzero idempotent in A. Then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||p||| = 1$.*

Proof. According to [29, Theorem 5], products of Peirce subspaces of A relative to p, as defined in Lemma 3.1, are included in the subspaces indicated in the table below:

	A_1	A_{10}	$A_{\frac{1}{2}\frac{1}{2}}$	A_{01}	A_0
A_1	A_1	A_{10}	$A_{\frac{1}{2}\frac{1}{2}}$	0	0
A_{10}	0	A_{01}	0	A_1	A_{10}
$A_{\frac{1}{2}\frac{1}{2}}$	$A_{\frac{1}{2}\frac{1}{2}}$	0	$A_1 + A_0$	0	$A_{\frac{1}{2}\frac{1}{2}}$
A_{01}	A_{01}	A_0	0	A_{10}	0
A_0	0	0	$A_{\frac{1}{2}\frac{1}{2}}$	A_{01}	A_0

(3.5)

As a first consequence, putting $A_2 := A_1 + A_0$, A_2 becomes a subalgebra of A, and the set $\{p\} \cup B_{A_1} \cup B_{A_0}$ is multiplicatively closed. Therefore, the absolutely convex hull of $\{p\} \cup B_{A_1} \cup B_{A_0}$ (say S) is bounded and multiplicatively closed, and contains εB_{A_2} for suitable $\varepsilon > 0$ (because, by Lemma 3.1, the direct sum $A_2 = A_1 \oplus A_0$ is topological). Since p lies in S, it follows from Lemma 2.2 that there exists an equivalent algebra norm $|||\cdot|||$ on A_2 such that $|||p||| = 1$. Moreover we claim that, for $i, j = 1, \frac{1}{2}, 0$ with $i + j = 1$, $x \in A_2$, and $y \in A_{ij}$, we have $\|xy\| \leqslant |||x|||\,\|y\|$ and $\|yx\| \leqslant \|y\|\,|||x|||$. Indeed, fixed i, j as above, the set

$$\{x \in A_2\colon\ \|xy\| \leqslant \|y\|\ \forall y \in A_{ij}\}$$

is absolutely convex and contains $\{p\} \cup B_{A_1} \cup B_{A_0}$, and hence also contains S. Since, by the proof of Lemma 2.2, $|||\cdot|||$ is the Minkowski functional of S on A_2, the above implies $\|xy\| \leqslant |||x||| \|y\|$ for all $x \in A_2$ and $y \in A_{ij}$. The inequality $\|yx\| \leqslant \|y\| |||x|||$ is proved in an analogous way.

Now we have $A = A_2 \oplus A_{10} \oplus A_{\frac{1}{2}\frac{1}{2}} \oplus A_{01}$, and, by (3.5), products of subspaces in this decomposition are included in the subspaces indicated in the table below:

	A_2	A_{10}	$A_{\frac{1}{2}\frac{1}{2}}$	A_{01}
A_2	A_2	A_{10}	$A_{\frac{1}{2}\frac{1}{2}}$	A_{01}
A_{10}	A_{10}	A_{01}	0	A_2
$A_{\frac{1}{2}\frac{1}{2}}$	$A_{\frac{1}{2}\frac{1}{2}}$	0	A_2	0
A_{01}	A_{01}	A_2	0	A_{10}

(3.6)

Take $M \geqslant 1$ such that $|||\cdot||| \leqslant M\|\cdot\|$ on A_2, and extend $|||\cdot|||$ to the whole algebra A by defining

$$|||x||| := |||x_2||| + M\big(\|x_{10}\| + \|x_{\frac{1}{2}\frac{1}{2}}\| + \|x_{01}\|\big)$$

for every $x = x_2 + x_{10} + x_{\frac{1}{2}\frac{1}{2}} + x_{01} \in A$. Since $|||\cdot|||$ is an equivalent norm on A_2, and the direct sum $A = A_2 \oplus A_{10} \oplus A_{\frac{1}{2}\frac{1}{2}} \oplus A_{01}$ is topological (again by Lemma 3.1), $|||\cdot|||$ is an equivalent norm on the vector space of A. Thus, the proof is concluded by showing that $|||\cdot|||$ is an algebra norm on A. Indeed, for $x = x_2 + x_{10} + x_{\frac{1}{2}\frac{1}{2}} + x_{01}$ and $y = y_2 + y_{10} + y_{\frac{1}{2}\frac{1}{2}} + y_{01}$ in A, we have

$$\begin{aligned}
|||xy||| &= |||x_2y_2 + x_{10}y_{01} + x_{\frac{1}{2}\frac{1}{2}}y_{\frac{1}{2}\frac{1}{2}} + x_{01}y_{10}||| \\
&\quad + M\big(\|x_2y_{10} + x_{10}y_2 + x_{01}y_{01}\| + \|x_2y_{\frac{1}{2}\frac{1}{2}} + x_{\frac{1}{2}\frac{1}{2}}y_2\| + \|x_2y_{01} + x_{10}y_{10} + x_{01}y_2\|\big) \\
&\leqslant |||x_2||| \, |||y_2||| + M\|x_{10}\|\|y_{01}\| + M\|x_{\frac{1}{2}\frac{1}{2}}\|\|y_{\frac{1}{2}\frac{1}{2}}\| + M\|x_{01}\|\|y_{10}\| \\
&\quad + M|||x_2|||\|y_{10}\| + M\|x_{10}\| |||y_2||| + M\|x_{01}\|\|y_{01}\| + M|||x_2|||\|y_{\frac{1}{2}\frac{1}{2}}\| \\
&\quad + M\|x_{\frac{1}{2}\frac{1}{2}}\| |||y_2||| + M|||x_2|||\|y_{01}\| + M\|x_{10}\|\|y_{10}\| + M\|x_{01}\| |||y_2||| \\
&\leqslant \big(|||x_2||| + M\big(\|x_{10}\| + \|x_{\frac{1}{2}\frac{1}{2}}\| + \|x_{01}\|\big)\big)\big(|||y_2||| + M\big(\|y_{10}\| + \|y_{\frac{1}{2}\frac{1}{2}}\| + \|y_{01}\|\big)\big) \\
&= |||x||| \, |||y|||.
\end{aligned}$$

In the equality at the beginning of the above computations we have kept in mind the definition of $|||\cdot|||$ on A, and (3.6). In the first inequality we have applied that $|||\cdot|||$ is an algebra norm on A_2 satisfying $|||\cdot||| \leqslant M\|\cdot\|$ on A_2, and the claim shown in the preceding paragraph. For the second inequality it is enough to remember that $M \geqslant 1$. $\square$

The following result is surely well known.

Lemma 3.3. *Let X and Y normed spaces, let v be a strongly exposed point of B_X, and let Z stand for the ℓ_1-sum of X and Y. Then v is a strongly exposed point of B_Z. If X and Y are dual Banach spaces, and if v is a w^*-strongly exposed point of B_X, then Z is a dual Banach space in a natural way, and v becomes a w^*-strongly exposed point of B_Z.*

Proof. We identify in the natural way Z^* with the ℓ_∞-sum of X^* and Y^*. Let v be strongly exposed by $g \in S_{X^*}$ in X. Let (z_n) be a sequence in B_Z such that $(g(z_n)) \to 1$. Then, writing $z_n = x_n + y_n$ with $(x_n, y_n) \in X \times Y$, we have $g(x_n) = g(z_n)$ and hence $(g(x_n)) \to 1$. Since $x_n \in B_X$, and v is strongly exposed by g in X, we deduce $(x_n) \to v$. On the other hand, we have

$$\|y_n\| = \|z_n\| - \|x_n\| \leqslant 1 - \|x_n\| \to 0.$$

It follows that $(z_n) \to v$. In this way, we have shown that v is strongly exposed by g in Z.

Assume that X and Y are dual Banach spaces (with preduals X_* and Y_*, respectively), and let v be strongly exposed by $g \in S_{X_*}$ in X. By the above paragraph, v is strongly exposed by g in Z. But Z identifies naturally with the dual of $Z_* := X_* \oplus_\infty Y_*$ (and hence is a dual Banach space), and g lies in Z_*. □

Corollary 3.4. *Let A be a normed generalized standard algebra, and let p be a nonzero idempotent in A. Then there exists an equivalent algebra norm $|||\cdot|||$ on A satisfying that $|||p||| = 1$ and that p is a strongly exposed point of $B_{(A,|||\cdot|||)}$. Moreover, if A is a dual Banach space in such a way that the operators of left and right multiplication by p on A become w^*-continuous, then the norm $|||\cdot|||$ above can be chosen among dual norms on A, and such that p becomes in fact a w^*-strongly exposed point of $B_{(A,|||\cdot|||)}$.*

Proof. First assume that $\|p\| > 1$. Let $|||\cdot|||$ the algebra norm on A_2 constructed in the proof of Theorem 3.2, so that $|||\cdot|||$ is an equivalent algebra norm on A_2 such that $|||p||| = 1$, and let S be also as in that proof. Since S is the absolutely convex hull of $\{p\} \cup [\mathrm{co}(B_{A_1} \cup B_{A_0})]$ (where $\mathrm{co}(\cdot)$ means convex hull), and $\mathrm{co}(B_{A_1} \cup B_{A_0})$ is the closed unit ball for a suitable norm $|\cdot|$ on A_2 such that $|p| = \|p\| > 1$, and $|||\cdot|||$ is the Minkowski functional of S on A_2, Lemma 2.7(2) applies giving us that p is a strongly exposed point of $B_{(A_2,|||\cdot|||)}$. Now, extend $|||\cdot|||$ to an algebra norm on the whole algebra A (as is done in the proof of Theorem 3.2), an apply Lemma 3.3 to deduce that p is a strongly exposed point of $B_{(A,|||\cdot|||)}$.

Now assume that $\|p\| = 1$. Then $2\|\cdot\|$ is an equivalent algebra norm on A whose value at p is strictly greater than 1, and the preceding paragraph applies.

Now assume that $\|p\| > 1$, and that A is a dual Banach space in such a way that L_p and R_p are w^*-continuous. Let $|||\cdot|||$ be the equivalent algebra norm on A_2 considered in the first paragraph of the proof, so that $|||p||| = 1$, and let S be as above, so that $S = \mathrm{co}[B_{A_1} \cup B_{A_0} \cup (B_{\mathbb{K}} p)]$. Note that, by (3.4), and the assumption that L_p and R_p are w^*-continuous, the projections from A onto A_1, A_{10}, $A_{\frac{1}{2}\frac{1}{2}}$, A_{01}, and A_0 corresponding to the decomposition $A = A_1 \oplus A_{10} \oplus A_{\frac{1}{2}\frac{1}{2}} \oplus A_{01} \oplus A_0$ are w^*-continuous, and hence A_1, A_{10}, $A_{\frac{1}{2}\frac{1}{2}}$, A_{01}, and A_0 are w^*-closed subspaces of A. As a first consequence, A_2 is w^*-closed in A, and both B_{A_1} and B_{A_0} are w^*-compact convex subsets of A. It follows that S is w^*-compact, and hence norm-closed in A. Keeping in mind that $|||\cdot|||$ is the Minkowski functional of S on A_2, we derive that $B_{(A_2,|||\cdot|||)} = S$, so $B_{(A_2,|||\cdot|||)}$ is w^*-closed in A_2, and so $|||\cdot|||$ is a dual norm on A_2. Moreover, arguing as in the first paragraph of the present proof, with Lemma 2.7(3) instead of Lemma 2.7(2), we realize that p is a w^*-strongly exposed point of $B_{(A_2,|||\cdot|||)}$ (note that, in our present situation, the norm $|\cdot|$ in the first paragraph is actually an equivalent dual norm on A_2). Now, extend $|||\cdot|||$ to an algebra norm on the whole algebra A (as is done in the proof of Theorem 3.2), an apply Lemma 3.3 to deduce that the extended norm (also denoted by $|||\cdot|||$) is a dual norm, and that p is a w^*-strongly exposed point of $B_{(A,|||\cdot|||)}$.

Finally assume that $\|p\| = 1$, and that A is a dual Banach space in such a way that L_p and R_p are w^*-continuous. Then $2\|\cdot\|$ is an equivalent dual algebra norm on A whose value at p is strictly greater than 1, and the preceding paragraph applies. □

Let A be an algebra over $\mathbb{K}$, and let λ be in $\mathbb{K}$. The λ-mutation of A is defined as the algebra whose vector space is that of A, and whose product (say $\square$) is defined by

$$a \,\square\, b := \lambda ab + (1 - \lambda)ba.$$

The class of noncommutative Jordan algebras is closed under mutations of its members, and hence contains all mutations of associative algebras. These last algebras are usually called split quasi-associative algebras. Now we can realize that Theorem 3.2 does not remain true if the assumption that A is generalized standard is relaxed to the one that A is noncommutative Jordan.

Example 3.5. Let λ be a real number with $\lambda > 1$. Then there exists a two-dimensional normed split quasi-associative algebra A with an idempotent p satisfying $\|p\| = \lambda$ and $|||p||| \geqslant \lambda$ for every algebra norm $|||\cdot|||$ on A. Indeed, let p and q be the elements of the associative algebra $M_2(\mathbb{K})$ (of all 2×2 matrices over $\mathbb{K}$) given by $p := \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $q := \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$, respectively, and let C denote the linear hull of $\{p,q\}$ in $M_2(\mathbb{K})$. Then C becomes a subalgebra of $M_2(\mathbb{K})$. Now take A equal to the λ-mutation of C, so that A is a two-dimensional split quasi-associative algebra, and the multiplication table of A is given by

$$\begin{array}{c|cc} & p & q \\ \hline p & p & \lambda q \\ q & (1-\lambda)q & 0 \end{array} \tag{3.7}$$

Clearly p becomes an idempotent in A. In addition, define a norm on A by $\|\alpha p+\beta q\| := \lambda|\alpha|+|\beta|$. It is easily realized that $\|\cdot\|$ becomes an algebra norm on A satisfying $\|p\| = \lambda$. Moreover, since $pq = \lambda q$ in A, for every algebra norm $|||\cdot|||$ on A we have $\lambda|||q||| = |||pq||| \leqslant |||p||| |||q|||$, and hence $\lambda \leqslant |||p|||$.

4. Some complements

Proposition 4.1. *Let A be a nonzero normed algebra with a unit $\mathbf{1}$. Then we have*:

(1) *If S is a bounded and multiplicatively closed subset of A contained in the nucleus of A, then there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||\mathbf{1}||| = 1$ and $|||s||| \leqslant 1$ for every $s \in S$.*

(2) *If $p \in A$ is a nuclear idempotent different from* 0 *and* $\mathbf{1}$, *then there exists an equivalent algebra norm $|||\cdot|||$ on A such that*

$$|||\mathbf{1}||| = |||p||| = |||\mathbf{1}-p||| = 1.$$

(3) *If A is generalized standard, and if $p \in A$ is an idempotent different from* 0 *and* $\mathbf{1}$, *then there exists an equivalent algebra norm $|||\cdot|||$ on A such that*

$$|||\mathbf{1}||| = |||p||| = |||\mathbf{1}-p||| = 1.$$

Proof. Let S be a bounded and multiplicatively closed subset of A contained in the nucleus of A. Then $S\cup\{\mathbf{1}\}$ is also a bounded and multiplicatively closed subset of A contained in the nucleus of A, and Theorem 2.3 applies.

Let $p \in A$ be a nuclear idempotent different from 0 and $\mathbf{1}$. Then $\{0, p, \mathbf{1}-p\}$ is a bounded and multiplicatively closed subset of A contained in the nucleus of A, so that (2) follows from (1).

The proof of (3) consists of a slight modification of the proof of Theorem 3.2. Assume that A is generalized standard, and let $p \in A$ be an idempotent different from 0 and $\mathbf{1}$. Let $A_1, A_{10}, A_{\frac{1}{2}\frac{1}{2}}, A_{01}, A_0$ be the subspaces of A introduced in Lemma 3.1, so that we have $p \in A_1$, $q := \mathbf{1}-p \in A_0$ (which implies $\mathbf{1} = p+q \in A_2 := A_1 + A_0$), and $qx = jx, xq = ix$ whenever $i, j = 1, \frac{1}{2}, 0$ with $i+j = 1$, and $x \in A_{ij}$. It follows from (3.5) that $A_2 := A_1 + A_0$ becomes a subalgebra of A, and that the set $\{\mathbf{1}, p, q\}\cup B_{A_1}\cup B_{A_0}$ is multiplicatively closed. Therefore, the absolutely convex hull of $\{\mathbf{1}, p, q\}\cup B_{A_1}\cup B_{A_0}$ (say T) is bounded and multiplicatively closed, and contains εB_{A_2} for suitable $\varepsilon > 0$ (because, by Lemma 3.1, the direct sum $A_2 = A_1 \oplus A_0$ is topological). Since $\{\mathbf{1}, p, q\} \subseteq T$, it follows from Lemma 2.2 that there exists an equivalent algebra norm $|||\cdot|||$ on A_2 such that $|||\mathbf{1}||| = |||p||| = |||q||| = 1$. Moreover, mimicing the corresponding part of the proof of Theorem 3.2 (when T replaces the set S in that proof), we realize that, for $i, j = 1, \frac{1}{2}, 0$ with $i+j = 1$, $x \in A_2$, and $y \in A_{ij}$, we have $\|xy\| \leqslant |||x||| \|y\|$ and $\|yx\| \leqslant \|y\| |||x|||$. By arguing as in the conclusion of the proof of Theorem 3.2, the fact just formulated allows us to extend $|||\cdot|||$ to an equivalent algebra norm on the whole algebra A. $\square$

It follows easily from the above proof that, if A is both a unital normed generalized standard algebra and a dual Banach space, and if $p \in A$ is an idempotent different from 0 and $\mathbf{1}$, then the equivalent algebra norm $|||\cdot|||$ on A given by Proposition 4.1(3) can be chosen among duals norms on A. Another result in the same direction is the following.

Proposition 4.2. *Let A be a normed algebra, and let p be a nonzero nuclear idempotent in A. Then there exists an equivalent algebra norm $|||\cdot|||$ on A satisfying that $|||p|||=1$ and that p is a strongly exposed point of $B_{(A,|||\cdot|||)}$. Moreover, if A is a dual Banach space in such a way that the operators of left and right multiplication by p on A become w^*-continuous, then the norm $|||\cdot|||$ above can be chosen among dual norms on A, and such that p becomes in fact a w^*-strongly exposed point of $B_{(A,|||\cdot|||)}$.*

Proof. Since p is a nuclear idempotent, we have $L_p^2 = L_p$, $R_p^2 = R_p$, and $L_pR_p = R_pL_p$. Therefore

$$P_{11} := L_pR_p,$$
$$P_{10} := L_p(1-R_p),$$
$$P_{01} := (1-L_p)R_p,$$
$$P_{00} := (1-L_p)(1-R_p)$$

are pair-wise orthogonal lineal projections on A the sum of which is equal to the identity on A. By putting $A_{ij} := P_{ij}(A)$ $(i,j=1,0)$, it follows that A has a Peirce decomposition (vector space direct sum)

$$A = \bigoplus_{i,j\in\{1,0\}} A_{ij}, \tag{4.1}$$

and that the above direct sum is topological. Moreover, if A is a dual Banach space in such a way that the operators L_p and R_p become w^*-continuous, then the subspaces A_{ij} are w^*-closed in A. On the other hand, involving again the assumption that the idempotent p is nuclear, we easily realize that

$$A_{ij}A_{kl} \subseteq \delta_{jk}A_{il} \tag{4.2}$$

for $i,j,k,l = 1,0$. Note also that

$$px = ix \quad \text{and} \quad xp = jx \quad (\text{for } i,j=1,0 \text{ and } x\in A_{ij}). \tag{4.3}$$

It follows from (4.2) and (4.3) that the set

$$S := \{p\} \cup B_{A_{11}} \cup B_{A_{10}} \cup B_{A_{01}} \cup B_{A_{00}}$$

is multiplicatively closed. Therefore, the absolutely convex hull of S is bounded and multiplicatively closed, and contains εB_A for suitable $\varepsilon > 0$ (because the direct sum (4.1) is topological). Since p lies in S, it follows from Lemma 2.2 that there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||p|||=1$. Now, if $\|p\|>1$ (and if A is a dual Banach space in such a way that the operators L_p and R_p become w^*-continuous), then we can argue as in the proof of Corollary 3.4 to obtain that p is a strongly exposed (w^*-strongly exposed) point of $B_{(A,|||\cdot|||)}$. If $\|p\|=1$, apply the above to the equivalent algebra norm $2\|\cdot\|$. □

Since one-dimensional algebras are associative, the two-dimensional nonassociative counterexamples to Corollary 1.2, given by Examples 2.1 and 3.5, are of the smallest possible dimension. However, counterexamples of such a kind cannot have a unit element because two-dimensional algebras with a unit are also associative. Therefore the following result has its own interest.

Proposition 4.3. *Let λ be a real number with $\lambda > 1$. Then there exists a three-dimensional commutative* (*respectively, split quasi-associative*) *normed algebra B with a unit $\mathbf{1}$ and an idempotent p such that $\|\mathbf{1}\| = 1$, $\|p\| = \lambda$, and $|||p||| \geqslant \lambda$ for every algebra norm $|||\cdot|||$ on B.*

Proof. Let A be the normed algebra given by Example 2.1 (respectively, Example 3.5). By taking B equal to the so-called "normed unital hull" of A, all properties asserted for B become obvious. We recall that the normed algebra B above is the vector space $\mathbb{K}\mathbf{1} \oplus A$ with product defined by

$$(\alpha_1\mathbf{1} + a_1)(\alpha_2\mathbf{1} + a_2) := \alpha_1\alpha_2\mathbf{1} + (\alpha_1 a_2 + \alpha_2 a_1 + a_1 a_2),$$

and norm defined by

$$\|\alpha\mathbf{1} + a\| := |\alpha| + \|a\|.$$

Let us also note that B contains A as a subalgebra, so that the restriction to A of any algebra norm on B becomes an algebra norm on A. □

5. Converting Theorem 1.1 into an axiom

We begin this section by realizing that associativity is not an essential requirement in Theorem 1.1.

Proposition 5.1. *Let A be a normed algebra such that*

$$\|a^2\| = \|a\|^2 \quad \textit{for every } a \in A, \tag{5.1}$$

and let S be a bounded and multiplicatively closed subset of A. Then we have that $\|s\| \leqslant 1$ for every $s \in S$.

Proof. Let s be in S. Define a sequence (s_n) in S by $s_1 := s$ and $s_{n+1} := s_n^2$. Then we have $\|s_n\| = \|s\|^{2^{n-1}}$ for every $n \in \mathbb{N}$. Since S is bounded (by M, say), we deduce that

$$\|s\| \leqslant M^{1/2^{n-1}} \to 1. \qquad \square$$

Let us say that a normed algebra A satisfies the norm square equality (in short, NSE), if the requirement (5.1) in the above proposition is fulfilled. Examples of normed algebras A satisfying the NSE are absolute-valued algebras, *JB*-algebras, and smooth-normed algebras. The standard references for these objects are [13,27], and [24, Section 3], respectively.

Absolute-valued algebras are defined as those real or complex algebras A endowed with a norm $\|\cdot\|$ satisfying $\|ab\| = \|a\|\|b\|$ for all $a, b \in A$. Among them we can find exactly three two-dimensional real nonassociative examples. These are the absolute-valued algebras whose normed spaces are equal to $\mathbb{C}$ (regarded as a real space), and whose products $\square$ are defined by either $\lambda \square \mu := \overline{\lambda\mu}$, $\lambda \square \mu := \bar{\lambda}\mu$, or $\lambda \square \mu := \lambda\bar{\mu}$ (see [27, p. 107]). Among absolute-valued algebras we also can find many real or complex infinite-dimensional examples which are "very nonassociative", in the sense that they do not satisfy any identity (see [27, Subsection 3.2]), as well as a unique real nonassociative example having a unit element. This last example is the eight-dimensional alternative algebra $\mathbb{O}$ of Cayley numbers (see [27, Theorem 2.1]).

JB-algebras are defined as those complete normed Jordan real algebras A satisfying $\|a\|^2 \leqslant \|a^2 + b^2\|$ for all $a, b \in A$. The nonassociative *JB*-algebra of smallest degree is the so-called three-dimensional spin factor. This *JB*-algebra (usually denoted by $\mathcal{S}_3$) is the vector space $\mathbb{R}^3$ with product defined by

$$(\lambda_1, \lambda_2, \lambda_3)(\mu_1, \mu_2, \mu_3) := (\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3, \lambda_1\mu_2 + \lambda_2\mu_1, \lambda_3\mu_1 + \lambda_1\mu_3),$$

and norm defined by

$$\|(\lambda_1,\lambda_2,\lambda_3)\| := |\lambda_1| + \sqrt{\lambda_2^2+\lambda_3^2}$$

(see [13, 2.9.6 and 2.9.7]). Clearly $\mathbf{1} := (1,0,0)$ is a unit element for $\mathcal{S}_3$.

Smooth-normed algebras are defined as those real normed algebras having a unit $\mathbf{1}$ satisfying $\|\mathbf{1}\| = 1$, and such that their normed spaces are smooth at $\mathbf{1}$. Their structure is well-understood. Indeed, as a consequence of [24, Theorem 27], they are noncommutative Jordan algebras, satisfy the NSE, and their normed spaces are pre-Hilbert spaces. Among them we only find three associative examples, namely $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, endowed with their absolute values. Moreover, as a consequence of [24, Proposition 24], given an arbitrary nonzero real pre-Hilbert space H, there exists exactly one commutative smooth-normed algebra whose normed space is H. The nonassociative smooth-normed algebra of smallest degree is the Euclidean space $\mathbb{R}^3$ with product defined by

$$(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3) := (\lambda_1\mu_1-\lambda_2\mu_2-\lambda_3\mu_3, \lambda_1\mu_2+\lambda_2\mu_1, \lambda_3\mu_1+\lambda_1\mu_3).$$

Remark 5.2. Let A be a normed algebra and let L be a Hausdorff locally compact topological space. Then the space $C_0(L,A)$ (of all A-valued continuous functions on L vanishing at infinity) becomes a normed algebra under the operations defined point-wise, and the sup norm. It is straightforward that $C_0(L,A)$ satisfies the NSE whenever A does. The NSE is also preserved by passing to ℓ_∞-sums of arbitrary length, and λ-mutations for $0 \leqslant \lambda \leqslant 1$ (note that, if $0 \leqslant \lambda \leqslant 1$, then the λ-mutation of any normed algebra A, with norm equal to that of A, becomes a normed algebra). In particular, starting from the absolute-valued algebra $\mathbb{H}$ of Hamilton's quaternions, we are provided with four-dimensional nonassociative split quasi-associative unital real normed algebras satisfying the NSE. All procedures collected in the present paragraph do not produce anything new in the class of *JB*-algebras, but they do in the class of absolute-valued algebras. On the other hand, the class of smooth-normed algebras is closed under λ-mutations of its members for $0 \leqslant \lambda \leqslant 1$, but the remaining procedures above, when applied to members of this class, produce objects outside the class. Note that the NSE is preserved by passing to subalgebras, and hence by passing to c_0-sums of arbitrary length. Note also that NSE is preserved by passing to normed ultraproducts (see [8] for the definition and properties).

Let us say that a normed algebra A satisfies the norm-one boundedness property (in short NBP) if, for every bounded and multiplicatively closed subset S of A, there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||s||| \leqslant 1$ for every $s \in S$. We know that associative normed algebras, absolute-valued algebras, *JB*-algebras, and smooth-normed algebras fulfill the NBP.

Proposition 5.3. *The NBP is preserved by passing to algebra equivalent renormings, subalgebras, and finite ℓ_∞-sums, but not by passing to quotients* (*by closed ideals*)*.*

Proof. That the NBP is inherited by algebra equivalent renormings and subalgebras is straightforward.

Let A and B be normed algebras satisfying the NBP, put $C := A \oplus_\infty B$, and let S be a bounded and multiplicatively closed subset of C. Denote by P_A and P_B the natural projections from C onto A and B, respectively. Then $P_A(S)$ and $P_B(S)$ are bounded and multiplicatively closed subsets of A and B, respectively. Since A and B satisfy the NBP, there exists an equivalent algebra norm $|||\cdot|||$ on each of them such that $|||P_A(s)||| \leqslant 1$ and $|||P_B(s)||| \leqslant 1$ for every $s \in S$. By putting $|||(a,b)||| := \max\{|||a|||, |||b|||\}$, $|||\cdot|||$ becomes an equivalent algebra norm on C such that $S \subseteq B_{(C,|||\cdot|||)}$.

The NBP is not preserved by passing to quotients because absolute-valued algebras fulfill the NBP, and every normed algebra is (isometrically algebra-isomorphic to) a quotient of a suitable absolute-valued algebra [27, Corollary 3.1]. □

Let A be an algebra, and let a be in A. The monomials on a are defined inductively, according to their “degree”. The unique monomial on a of degree 1 is a, and, for $1 < n \in \mathbb{N}$, the monomials on a of degree n are those elements of A which can be written as xy with x and y monomials on a of degree $i \in \mathbb{N}$ and $j \in \mathbb{N}$, respectively, with $i + j = n$. Now assume that A is normed. For n in $\mathbb{N}$, let $M_n(a)$ stand for the maximum of the values of the norm at all monomials on a with degree equal to n, note that $M_n(a) \leqslant \|a\|^n$, and define the spectral radius $r(a)$ of a by

$$r(a) := \limsup_{n\to\infty}\bigl\{M_n(a)^{\frac{1}{n}}\colon\ n \in \mathbb{N}\bigr\} \leqslant \|a\|.$$

It is easily realized that equivalent algebra norms on A give the same spectral radius for a, and hence that $r(a) \leqslant |||a|||$ for every equivalent algebra norm $|||\cdot|||$ on A. Moreover, if the subalgebra of A generated by a is associative, then the spectral radius of a just defined coincides with the classical spectral radius $\lim_{n\to\infty}\{\|a^n\|^{\frac{1}{n}}\}$. Now we have the following.

Proposition 5.4. *Let A be a normed algebra satisfying the NBP, and let a be in A. Then we have*

(1) $r(a) = \inf\{|||a|||\colon\ |||\cdot||| \in \mathrm{En}(A)\}$.
(2) *If A has a unit $\mathbf{1}$, then we have in fact*

$$r(a) = \inf\bigl\{|||a|||\colon\ |||\cdot||| \in \mathrm{Eun}(A)\bigr\}.$$

Proof. To prove (1), it is enough to show that, if $r(a) < 1$, then there exists $|||\cdot||| \in \mathrm{En}(A)$ such that $|||a||| \leqslant 1$. But, if $r(a) < 1$, then we have $M_n(a) < 1$ for n big enough, and hence the set of all monomials on a (say S) is bounded. Since S is multiplicatively closed, the assumed NBP of A concludes the proof.

The proof of (2) is similar, and is left to the reader. □

Corollary 5.5. *Let A be a normed algebra over $\mathbb{K}$ satisfying the NBP, let a be in A, and let α and β be in $\mathbb{K}$. Then we have*

$$r(\alpha L_a + \beta R_a) \leqslant \bigl(|\alpha| + |\beta|\bigr)r(a).$$

Proof. For every equivalent algebra norm $|||\cdot|||$ on A, we have

$$r(\alpha L_a + \beta R_a) \leqslant |||\alpha L_a + \beta R_a||| \leqslant \bigl(|\alpha| + |\beta|\bigr)|||a|||.$$

Now apply Proposition 5.4. □

By a nearly absolute-valued algebra we mean a normed algebra A such that there exists $\rho > 0$ satisfying $\|ab\| \geqslant \rho\|a\|\|b\|$ for all $a, b \in A$. The class of nearly absolute-valued algebras is closed under algebra equivalent renormings of its members, and hence contains the class of all algebra equivalent renormings of absolute-valued algebras. However, there are nearly absolute-valued algebras which are not algebra equivalent renormings of absolute-valued algebras. The examples of such a pathology, pointed out until now, are the λ-mutations of $\mathbb{H}$ and $\mathbb{O}$ for $\frac{1}{2} < \lambda < 1$, and the $\frac{1}{2}$-mutations of certain infinite-dimensional real or complex absolute-valued algebras (see [15] and [27, Subsection 4.1]). Keeping in mind that these examples satisfy the NSE, and hence the NBP, the next result shows the existence of nearly absolute-valued algebras which are even much more perverse.

Proposition 5.6. *Let λ be a real number with $\lambda > 1$. Then there exists a four-dimensional split quasi-associative nearly absolute-valued real algebra A with a unit $\mathbf{1}$ satisfying $\|\mathbf{1}\| = 1$, and an element $a \in A$ such that $a^2 = -\mathbf{1}$, $\|a\| = \lambda$, and $r(L_a) = \lambda$. As a consequence, we have:*

(1) $r(a) = 1$
(2) $|||a||| \geqslant \lambda$ *for every algebra norm $|||\cdot|||$ on A.*

(3) *A does not satisfy the NBP.*
(4) *A does not satisfy the NSE under any algebra renorming.*

Proof. Take a canonical basis $\{\mathbf{1}, i, j, k\}$ of $\mathbb{H}$ as in [7, Definition I.14.3], so that we have $i^2 = -\mathbf{1}$, $ij = -ji = k$, and

$$|\mathbf{1}| = |i| = |j| = |k| = 1,$$

where $|\cdot|$ denotes the absolute-value of $\mathbb{H}$. Now, take A equal to the $\frac{\lambda+1}{2}$-mutation of $\mathbb{H}$, with algebra norm $\|\cdot\|$ to be determined immediately below, and put $a := i$. Since $\lambda|\cdot|$ is an algebra norm on A, we can apply Corollary 2.4 and equality (2.3) to obtain the desired algebra norm $\|\cdot\|$ on A, which satisfies $\|\mathbf{1}\| = 1$ and $|\cdot| \leqslant \|\cdot\| \leqslant \lambda|\cdot|$. This implies $\|a\| \leqslant \lambda$ and $\|xy\| \geqslant \lambda^{-2}\|x\|\|y\|$ for all $x, y \in A$, where now the juxtaposition means the product of A. Although $|\cdot|$ is not an algebra norm on A, the corresponding operator norm (also denoted by $|\cdot|$) is an algebra norm on the algebra of all linear operators on A, and hence can be used to compute the spectral radius of the operator L_a of left multiplication by a on A. Indeed, regarded as an operator on the Hilbert space $(A, |\cdot|)$, we have $L_a^* = -L_a$, and hence $r(L_a) = |L_a|$. But the inequality $|L_a| \leqslant \lambda$ is straightforward, and, keeping in mind that $L_a(j) = \lambda k$, the converse one is verified as follows:

$$\lambda = |\lambda k| = \left|L_a(j)\right| \leqslant |L_a||j| = |L_a|.$$

Assertion (1) follows from the fact that $a^2 = -\mathbf{1}$.

Let $|||\cdot|||$ be an equivalent algebra norm on A. Since $r(L_a) = \lambda$, we have $\lambda \leqslant |||L_a||| \leqslant |||a|||$, which proves (2).

Assertion (3) follows from (1), (2), and Proposition 5.4.

Finally, Assertion (4) follows from (3) and Proposition 5.1. □

For any nonzero normed algebra A, let $\rho(A)$ denote the largest nonnegative real number ρ such that the inequality $\|ab\| \geqslant \rho\|a\|\|b\|$ holds for all $a, b \in A$, so that A is absolute-valued (respectively, nearly absolute-valued) if and only if $\rho(A) = 1$ (respectively, $\rho(A) > 0$). It follows from Proposition 5.6 and its proof that the NBP fails for suitable nonzero normed algebras A with $\rho(A)$ arbitrarily close to 1.

Let A be an algebra. We define the annihilator of A by the equality

$$\operatorname{Ann}(A) := \{a \in A \colon aA = Aa = 0\}.$$

Clearly, $\operatorname{Ann}(A)$ is an ideal of A, and, in the case that A is normed, $\operatorname{Ann}(A)$ is closed in A.

Lemma 5.7. *Let A be a normed algebra such that $A/\operatorname{Ann}(A)$ satisfies the NBP. Then A fulfils the NBP.*

Proof. Let $\pi : A \to A/\operatorname{Ann}(A)$ stand for the natural surjection. For $a, b \in A$ and $x, y \in \operatorname{Ann}(A)$, we have $ab = (a + x)(b + y)$, and hence $\|ab\| \leqslant \|a + x\|\|b + y\|$, which implies

$$\|ab\| \leqslant \left\|\pi(a)\right\|\left\|\pi(b)\right\|. \tag{5.2}$$

Now, let S be a bounded an multiplicatively closed subset of A. Then $\pi(S)$ is a bounded and multiplicatively closed subset of $A/\operatorname{Ann}(A)$, so that, since $A/\operatorname{Ann}(A)$ satisfies the BNP, there exists an equivalent algebra norm $|\cdot|$ on $A/\operatorname{Ann}(A)$ such that $|\pi(s)| \leqslant 1$ for every $s \in S$. Take $\delta > 0$ such that $\delta\|\cdot\| \leqslant |\cdot|$ on $A/\operatorname{Ann}(A)$ and $\delta^2\|\cdot\| \leqslant 1$ on S, and for $a \in A$ put $|||a||| := \max\{\delta^2\|a\|, |\pi(a)|\}$. Keeping in mind (5.2), we easily realize that $|||\cdot|||$ is an equivalent algebra norm on A such that $|||\cdot||| \leqslant 1$ on S. □

Let A be an algebra. We say that A is nilpotent if there exists a natural number $n \geqslant 2$ such that any product of n elements of A, no matter how associated, is 0. The smallest such an n is called the index of nilpotence of A. As a consequence of [28, Theorem 2.4], if A is commutative or anti-commutative, then A is nilpotent if and only if there is $m \in \mathbb{N}$ such that $L_{a_1} L_{a_2} \dots L_{a_m} = 0$ whenever $a_1, a_2, \dots, a_m$ are in A.

Proposition 5.8. *Let A be a nilpotent normed algebra. Then A satisfies the NBP.*

Proof. We argue by induction on the index of nilpotence of A (say n). If $n = 2$, then the product of A is identically 0, so all norms on A are algebra norms, and so, given any bounded subset S of A, it is enough to multiply the norm of A by a suitable positive number to obtain an equivalent algebra norm on A whose values at elements of S are $\leqslant 1$. Assume that $n > 2$, and that all nilpotent normed algebras of index of nilpotence less than n satisfy the NBP. Let $\pi : A \to A/\mathrm{Ann}(A)$ stand for the natural surjection. If x is a product of $n-1$ elements of A, no matter how associated, then x lies in $\mathrm{Ann}(A)$, and hence $\pi(x) = 0$. Since π is a surjective algebra homomorphism, the above means that $A/\mathrm{Ann}(A)$ has index of nilpotence less than or equal to $n-1$. By the induction hypothesis, $A/\mathrm{Ann}(A)$ fulfils the NBP. By Lemma 5.7, A also fulfils the NBP. $\square$

We recall that an element a of an algebra A over a field K is said to be algebraic if the subalgebra of A generated by a is finite-dimensional.

Lemma 5.9. *Let A be an associative normed algebra with a unit, and let a be an algebraic element of A such that $r(a) = 0$. Then a is nilpotent.*

Proof. Denote by $\mathbb{K}[a]$ the subalgebra of A generated by $\{\mathbf{1}, a\}$. Since a is algebraic, there exists an idempotent $p \in \mathbb{K}[a]$ such that $a(\mathbf{1} - p)$ is nilpotent, and ap is invertible in $\mathbb{K}[a]p$ (see [12, Lemma 3.2]). Now it is enough to show that $p = 0$. But, if p were not 0, then, denoting by b the inverse of ap in $\mathbb{K}[a]p$, for every $n \in \mathbb{N}$ we would have

$$\|p\| = \left\|pa^n b^n\right\| \leqslant \|p\| \left\|a^n\right\| \left\|b^n\right\|,$$

which would imply $1 \leqslant r(a)r(b)$, contradicting the assumption that $r(a) = 0$. $\square$

An algebra A is said to be algebraic if, for very $a \in A$, the operators L_a and R_a are algebraic. This notion of algebraicity does not coincide with that of A.A. Albert [2] that every element of A generates a finite-dimensional subalgebra. Indeed, Albert's notion of algebraicity trivializes in the class of anti-commutative algebras, as is indeed useless in that class.

Proposition 5.10. *Let A be an anti-commutative normed algebra satisfying the NBP. Then $r(L_a) = 0$ for every $a \in A$. Moreover we have*:

(1) *If A is algebraic, then, for every $a \in A$, the operator L_a is nilpotent.*
(2) *If A is algebraic and complete, then there exists $n \in \mathbb{N}$ such that $L_a^n = 0$ for every $a \in A$.*

Proof. Since A is anti-commutative, we have $a^2 = 0$, and hence $r(a) = 0$. Keeping in mind that A fulfils the NBP, it follows from Corollary 5.5 that $r(L_a) = 0$.

Assume that A is algebraic. Then, by the above and Lemma 5.9, we conclude that L_a is nilpotent for every $a \in A$.

Assume that A is algebraic and complete. Then, invoking the Baire category theorem, and arguing as in the proof of [7, Theorem VII.46.3], we deduce from (1) the existence of some $n \in \mathbb{N}$ such that $L_a^n = 0$ for every $a \in A$. $\square$

Lie algebras are defined as those anti-commutative algebras A satisfying the Jacobi identity $(ab)c + (bc)a + (ca)b = 0$ for all $a, b, c \in A$. All associative algebras become Lie algebras under the commutator product

$$(a, b) \to ab - ba,$$

and there are no Lie algebras others than the subalgebras of the ones obtained by this procedure (see [29, p. 3]). We note that every associative normed algebra A becomes a Lie normed algebra (denoted by A^-) under the commutator product and the norm $2\|\cdot\|$.

A celebrated theorem of E.I. Zel'manov [30] asserts that a Lie algebra A over a field of characteristic zero is nilpotent whenever there exists $n \in \mathbb{N}$ such that $L_a^n = 0$ for every $a \in A$. Therefore, by putting together Propositions 5.8 and 5.10, we obtain the following.

Theorem 5.11. *Let A be a complete normed algebraic Lie algebra. Then A satisfies the NBP if and only if A is nilpotent.*

In relation to the above theorem, we note that, according to [9, Theorem A], a complete normed Lie algebra A is algebraic whenever it is "weakly algebraic", which means that, for each $(a, b) \in A \times A$ there exists a nonzero polynomial P such that $P(L_a)(b) = 0$.

Let A be a finite-dimensional algebra over $\mathbb{K}$. We know that there are always algebra norms on A, and that all these norms are equivalent. Since the NBP does not depend on the chosen algebra norm, we will not emphasize that A is normed when we are discussing about the NBP on A. This convention is applied in the statement of the following straightforward consequence of Theorem 5.11.

Corollary 5.12. *Let A be a finite-dimensional Lie algebra over $\mathbb{K}$. Then A satisfies the NBP if and only if A is nilpotent.*

The typical example of a finite-dimensional nilpotent Lie algebra over $\mathbb{K}$ is the subalgebra of $M_n(\mathbb{K})^-$ consisting of all strictly triangular matrices (i.e., triangular matrices with zero diagonal). Such Lie algebra is not associative whenever $n \geqslant 4$.

6. Concluding remarks, and questions

6.1. We do not know whether standard generalized normed algebras satisfy the NBP, nor even whether alternative or Jordan normed algebras do satisfy it. Given a field K, the basic examples of alternative nonassociative algebras over K are the so-called Cayley algebras over K. We refer the reader to [29, Sections III.4 and III.5] for the definition and properties of such algebras, limiting ourselves to point out that they are eight-dimensional over K, and that, among them, there is exactly one which has divisors of zero. Such an algebra is called the split Cayley algebra over K, and is denoted by $\mathcal{U}_K$. According to [29, Lemma 3.16], we have $\mathcal{U}_K = M_2(K) \oplus vM_2(K)$ with product given by

$$(x_1 + vx_2)(y_1 + vy_2) := (x_1y_1 + y_2\bar{x}_2) + v(\bar{x}_1y_2 + y_1x_2),$$

where, for $x = \left(\begin{smallmatrix} \lambda & \mu \\ \alpha & \beta \end{smallmatrix}\right) \in M_2(K)$, $\bar{x}$ is defined by $\bar{x} := \left(\begin{smallmatrix} \beta & -\mu \\ -\alpha & \lambda \end{smallmatrix}\right)$. If K is algebraically closed, then $\mathcal{U}_K$ is indeed the unique Cayley algebra over K. Moreover, there are exactly two real Cayley algebras, the non-split one being the division algebra $\mathbb{O}$, already introduced. We know that $\mathbb{O}$ has the NBP, but we do not know whether this is the case for $\mathcal{U}_{\mathbb{R}}$ or $\mathcal{U}_{\mathbb{C}}$.

6.2. Let A be a normed algebra satisfying the NSE. Then, by Proposition 5.1, B_A is the largest bounded and multiplicatively closed subset of A. As a consequence, we have $\|\cdot\| \leqslant |||\cdot|||$ for every $|||\cdot||| \in \mathrm{En}(A)$. Since the NSE implies the NBP, Proposition 5.4 applies, giving that $\|a\| \leqslant r(a)$ for every $a \in A$. Since the converse inequality is obvious, we actually have $\|a\| = r(a)$ for every $a \in A$. This implies that continuous algebra homomorphisms from arbitrary normed algebras to A are contractive

[27, Proposition 1.1]. Indeed, it is easy to realize that continuous algebra homomorphisms between normed algebras are contractive relative to the spectral radius.

6.3. We do not know "natural" examples of nonassociative normed complex algebras satisfying the NSE others than all infinite-dimensional absolute-valued complex algebras, and those nonassociative normed algebras obtained from them thanks to the stability properties of the NSE (see Remark 5.2). Note that these last algebras remain infinite-dimensional as well. This scarcity of such examples is not casual. Indeed, $\mathbb{C}$ is the unique finite-dimensional absolute-valued complex algebra (see [27, Subsection 2.8]). On the other hand, if a normed noncommutative Jordan complex algebra satisfies the NSE, then it is associative and commutative [23, Lemma 30], and hence (by Gelfand theory) is isometrically algebra-isomorphic to a subalgebra of $C_0(L,\mathbb{C})$ for some Hausdorff locally compact topological space L. Note also that smooth-normed complex algebras could have been defined verbatim as in the real case, but such a definition would become useless because $\mathbb{C}$ would be the unique smooth-normed complex algebra [24, Section 3].

Despite the above comments, exotic examples of finite-dimensional nonassociative normed complex algebras satisfying the NSE can be built by keeping in mind that, if a normed algebra A satisfies the NSE, and if ϕ and ψ are linear isometries from A to A, then the normed algebra whose normed space is that of A, and whose product is given by $(a,b)\to\phi(\psi(a)\psi(b))$, also satisfies the NSE. As an application, the normed algebra over $\mathbb{K}$ whose vector space is $\mathbb{K}^2$, whose norm is the ℓ_∞-norm, and whose product is defined by

$$(\lambda_1,\lambda_2)(\mu_1,\mu_2):=(\lambda_2\mu_2,\lambda_1\mu_1),$$

is not associative and fulfils the NSE.

6.4. Let A be a normed algebra over $\mathbb{K}$, and let a be in A. According to Corollary 5.5, if A satisfies the NBP, then we have

$$r(\alpha L_a+\beta R_a)\leqslant\big(|\alpha|+|\beta|\big)r(a)\quad\text{for all }\alpha,\beta\in\mathbb{K},\tag{6.1}$$

and hence

$$\max\big\{r(L_a),r(R_a)\big\}\leqslant r(a).\tag{6.2}$$

On the other hand, if A is flexible, then Inequalities (6.1) and (6.2) are in fact equivalent because in this case the operators L_a and R_a commute, and therefore the spectral radius is subaditive on the linear subspace generated by them. Inequality (6.2) is fulfilled whenever A is alternative (because, for every $n\in\mathbb{N}$, we have $L_{a^n}=L_a^n$ and $R_{a^n}=R_a^n$ in this case) or Jordan (as a consequence of [19, Theorem 1.2], after passing to complexification and completion if necessary), so, most probably, it is fulfilled also whenever A is standard generalized. Since standard generalized algebras are flexible, with the same probability, Inequality (6.1) also holds in the case that A is standard generalized. We recall that we do not know whether standard generalized normed algebras satisfy the NBP. By looking at Example 3.5 or Proposition 5.6, we realize that Inequality (6.2) (and hence (6.1)) need not remain true if A is merely assumed to be a noncommutative Jordan algebra. Finally note that, if A is power-associative, then the inequality

$$r(a)\leqslant\min\big\{r(L_a),r(R_a)\big\}$$

holds (since for every $n\in\mathbb{N}$ we have $L_a^n(a)=a^{n+1}$ and $R_a^n(a)=a^{n+1}$ in this case), and therefore, in this case, inequality (6.2) is equivalent to

$$r(a)=r(L_a)=r(R_a).$$

6.5. According to Proposition 5.10, if A is an anti-commutative normed algebra satisfying the NBP, then we have $r(L_a) = 0$ for every $a \in A$. This means that "most" anti-commutative normed algebras fail the NBP.

For instance, if X is any normed space over $\mathbb{K}$ with $\dim(X) \geqslant 2$, and if $\mathcal{L}(X)$ stand for the associative normed algebra of all bounded linear operators on X, then the Lie normed algebra $\mathcal{L}(X)^-$ fails the NBP. Indeed, taking a bounded linear projection P onto some two-dimensional subspace of X, $P\mathcal{L}(X)P$ becomes a subalgebra of $\mathcal{L}(X)$ isomorphic to $M_2(\mathbb{K})$, so that it is enough to show that $M_2(\mathbb{K})^-$ fails the NBP. But this follows for example by noticing that, for $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $pq - qp = q$, which implies that the spectral radius of the left multiplication by p on $M_2(\mathbb{K})^-$ is not zero.

As another application of the fact that $r(L_a) = 0$ whenever a is any element of an anti-commutative normed algebra satisfying the NBP, we realize that anti-commutative normed complex algebras "with hermitian multiplication" (in the sense of [25, p. 10]) fail the NBP, as soon as they have nonzero product. Indeed, this follows easily from the definition of such algebras and [7, Theorem I.10.17]. Among anti-commutative normed complex algebras with hermitian multiplication and nonzero product we find all noncommutative C^*-algebras regarded as Lie–Banach algebras, all anti-commutative complex H^*-algebras with nonzero product (see [26, Section E]), and the Lie–Banach algebra of all derivations of an arbitrary noncommutative C^*-algebra [25, Remark 2.8.(iii)].

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On multiplicatively closed subsets of normed algebras. Journal of Algebra, Vol. 323, Issue. 5, p. 1530. CrossRef.Â It is proved that every nonempty relatively weakly open subset of the closed unit ball B_A of A has diameter equal to 2. This implies that B_A is not dentable, and that there is not any point of continuity for the identity mapping $(B_A,{\rm weak})\,{\longrightarrow}\,(B_A,{\rm norm})$. Export citation. On multiplicatively closed subsets of normed algebras. Article. Mar 2010.Â We also show that, if an anti-commutative complete normed algebraic algebra A satisfies the NBP, then there exists nâˆˆN such that Lan=0 for every aâˆˆA, where La denotes the operator of left multiplication by a. It follows from a celebrated theorem of E.I. Zel'manov on the so-called Engel Lie algebras that a complete normed. In abstract algebra, a multiplicatively closed set (or multiplicative set) is a subset S of a ring R such that the following two conditions hold:[1][2]. 1 âˆˆ S {\displaystyle 1\in S}. . For all x and y in S, the product xy is in S. In other words, S is closed under taking finite products, including the empty product 1.[3] Equivalently, a multiplicative set is a submonoid of the multiplicative monoid of a ring. Multiplicative sets are important especially in commutative algebra, where they are used to build localizations of commutative rings. A subset S of a ring R is called saturated if it is c...