A Survey of
“Complexity Measures”

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The term *complexity* has many different meanings. At least one adjective is needed to help distinguish between different uses of the word:

- Kolmogorov-Chaitin Complexity
- Computational Complexity
- Stochastic Complexity
- Statistical Complexity
- Structural Complexity
Deterministic Complexity

The *Kolmogorov-Chaitin* complexity $K(x)$ of an object $x$ is the length, in bits, of the smallest program (in bits) that when run on a *Universal Turing Machine* outputs $x$ and then halts.

References:


**Books:**

Measures of Randomness

The entropy rate $h_\mu$ of a symbolic sequence measures the unpredictability (in bits per symbol) of the sequence. The entropy rate is also known as the entropy density or the metric entropy.

References:

- Boltzmann (1866).
- Books:
Kolmogorov Complexity $\approx$ Randomness!!

The Kolmogorov complexity $K(x)$ is maximized for random strings.

The average growth rate of $K(x)$ is equal to the entropy rate $h_\mu$.

If $x = \text{trajectory of a chaotic dynamical system } f$:

$$K(x(t)) = h_\mu(f) \quad \text{for typical } x(0).$$


If a string $x$ is random, then it possesses no regularities. Thus,

$$K(x) = |\text{Print}(x)|.$$

That is, the shortest program to get a UTM to produce $x$ is to just hand the computer a copy of $x$ and say “print this.”
Measures of “Complexity” that Capture a Property Distinct from Randomness

The entropy rate $h_\mu$ and the Kolmogorov Complexity $K(x)$ do not measure pattern or structure or correlation or organization.

Structure or pattern is maximized for neither high nor low randomness.

Note: The structural complexity vs. randomness relation above is just one of many possible behaviors. Different systems have different structural complexity vs. randomness plots. There is no “universal” complexity-entropy relationship! (E.g., Feldman and Crutchfield, Phys. Lett. A, 238:244-252, (1998), and references therein.)
Information Theoretic Approaches to Structural Complexity

Entropy Density Convergence and/or Mutual Information:

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Notes on Terminology

All of the following terms refer to (essentially) the same quantity.

- **Excess Entropy**: Crutchfield, Packard, Feldman
- **Stored Information**: Shaw
- **Effective Measure Complexity**: Grassberger, Lindgren, Nordahl
- **Reduced (Rényi) Information**: Szépfalusz, Györgyi, Csordás
- **Complexity**: Li, Arnold
Early Uses of Mutual Information


Computational Mechanics

Discover and Quantify Structure by Using a Combination of Computation Theory, Statistical Inference, and Information Theory

Computational Mechanics seeks to Detect the Intrinsic Computation being Performed by the System

- Crutchfield and Young, in *Complexity, Entropy and the Physics of Information*, Addison-Wesley, 1990.
Other Approaches to Complexity Distinct from Randomness

Logical Depth:

The **Logical Depth** of $x$ is the **run time** of the shortest program that will cause a UTM to produce $x$ and then halt.

Logical depth is not a measure of randomness; it is small for both trivially ordered and random strings.

References:


Thermodynamic Depth:

Proposed as a measure of structural complexity (Lloyd and Pagels, *Annals of Physics*, 188:186-213). However, thermo. depth depends crucially on the choice of state. Lloyd and Pagels give no general prescription for how states should be chosen. Once states are chosen, thermo. depth is equivalent to the reverse time entropy rate. (Shalizi and Crutchfield, (in preparation), 1998.)
Other Approaches to Complexity Distinct from Randomness

Sophistication:

Reference:


Effective Complexity:

Reference:

Other Approaches to Complexity Distinct from Randomness

Non-Linear Modeling

References:

- Wallace and Boulton, 1968.
- Crutchfield and Young, in *Complexity, Entropy and the Physics of Information*, Addison-Wesley, 1990.

Model Convergence and Hierarchical Grammatical Complexities:

References:


Note: Badii and Politi’s book contains a solid discussion of many different structural complexity measures.
Other Approaches to Complexity Distinct from Randomness

Miscellaneous References:

Non-constructive Complexity Measures: the Road Untakable

All Universal Turing Based complexity measures suffer from several drawbacks:

1. They are uncomputable.

2. By adopting a UTM, the most powerful discrete computation model, one loses the ability to distinguish between systems that can be described by computational models less powerful than a UTM.

UTM-based “complexity” measures include:


Falling off the “Edge of Chaos”

• Packard, ”Adaptation to the Edge of Chaos” in Dynamic Patterns in Complex Systems, Kelso et.al, eds., World Scientific, 1988


• Wooters and Langton, ”Is there a Sharp Phase Transition for Deterministic Cellular Automata?”, Physica D 45 (1990) 95.

This paper surveys and analyzes measures which can be extracted from the training datasets in order to characterize the complexity of the respective classification problems. Their use in recent literature is also reviewed and discussed, allowing to prospect opportunities for future work in the area. Finally, descriptions are given on an R package named Extended Complexity Library (ECoL) that implements a set of complexity measures and is made publicly available. Discover the world's research. 15+ million members. Particularly, code complexity measures have been designed to measure and monitor complexity in practice (Zuse 1991; Abran 2010; Fenton and Bieman 2014). Complexity measurement allows quantifying complexity and understanding its effect on maintainability and defect proneness. The concept of code complexity, however, is not an atomic concept, so it is difficult to design a single measure that quantifies code complexity thoroughly. Instead, several complementary measures are designed to measure different aspects of complexity. At the end of this part of the survey, an open question was included to allow respondents to suggest other code characteristics that they believed could significantly increase complexity. 3.3 Complexity and Internal Code Quality Attributes. indirect measure of well complexity, since complex wells are frequently associated with multiple strings, and narrow margins between formation pore pressure and fracture pressure gradients often result in the requirement of a greater number of. The key drilling factors are user-defined qualitative variables %i that are assigned an integer-valued weight ψ_i (w) according to the occurrence of the condition and its degree of complexity. Let ψ_i denote the ith drilling factor of well w and ψ_i (w) the corresponding numerical weight.