NONUNITARY LIE-ISOTOPIC AND LIE-ADMISSIBLE SCATTERING THEORIES OF HADRONIC MECHANICS, II: Deformations-Isotopies of Lie's Theory, Special Relativity, and Mechanics

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Abstract

In the preceding Paper I, we have presented a variety of aspects suggesting a reinspection of the elaboration of measured quantities (cross section, scattering angle, etc.) via the conventional unitary scattering theory due to possible non-Hamiltonian internal effects implying a nonunitary time evolution. We have then reviewed the inconsistency theorems for nonunitary theories on conventional spaces over conventional fields, outlined the foundations of the novel isomathematics permitting a resolution of said inconsistency theorems, and suggested an isounitary reformulation of nonunitary scattering theories. In this paper we outline the use of isomathematics to achieve of methods essential for a consistent treatment of nonunitary-isounitary theories for interior dynamical conditions, such as the deformations-isotopies of Lie's theory, special relativity and mechanics. The outline appears recommendable due to a variety of formulations existing in the literature often leading to misconceptions because of their inapplicability to scattering problems, or formulations prior to the resolution of the inconsistency theorems. Following this necessary background, the formulation of the isoscattering theory without divergencies *ab initio* will be presented in Paper III, and comparative data elaborations via the conventional and the isotopic scattering theory will be initiated in Paper IV. ..

Key words: Lie algebnras, special relativity, Hamiltonian mechanics

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1. Deformations-Isotopies of Lie's Theory

1.1. Introduction. In the preceding paper [1], hereinafter referred to as Paper I, we have presented rather diversified conceptual, theoretical and experimental elements suggesting a reinspection of the validity of special relativity for interior dynamical problems at large, and the scattering region in particular.

Consequently, the central problem in applied mathematical underlying the proposed isoscattering theory is the construction of a covering of the Minkowskian geometry, the Lorentz-Poincaré symmetry and special relativity into forms more effective for interior conditions.

In turn, the study of the above problem requires the achievement of the universal invariance of interior systems with a locally varying speeds of light

$$C = \frac{c}{n(x, v, \xi, \omega, \psi, \partial\psi, ...)} \tag{1.1}$$

where c is the speed of light in vacuum, and $n(x, v, \xi, \omega, \psi, \partial\psi, ...)$ is the index of refraction generally dependent on local coordinates x, velocities v (e.g., of the medium with respect to the source), density of the medium ξ, frequency of light ω, wavefunction ψ, its derivative $\partial\psi$,and other variables. We assume the reader is aware from Paper I that local speed (1.1) is assumed in the isoscattering theory as applying also to photons, since they cannot be assumed, without due inspection, as propagating in vacuum when in the interior of the scattering region due to its hyperdense character.

In view of the primitive character of light for all of physics, the study of the isoscattering theory can be reduced to the study of photons propagating within a hyperdense scattering region composed of particles in conditions of total mutual penetration. In the event the elaboration of measured quantities (cross section, scattering angles, etc.) via the isoscattering theory turns out to be entirely equivalent to the conventional elaboration, photons within the scattering regions would be confirmed as propagating in vacuum with consequential full validity of special relativity. By contrast, possible differences in the two data elaborations would establish that photons within the scattering region have local speed (1.1) with consequential need for a covering spacetime geometry, symmetry and relativity.

1.2. The Forgotten Lorentz Problem. Since the speed of light during pre-Einstein's time was considered to be a local quantity $C = c/n$, Lorentz [2] studied its invariance, as noted by Pauli in his celebrated book *Theory of Relativity,* but encountered major technical difficulties for the case of the index of refraction with an arbitrary functional dependence (rather than constant) and had to restrict his studies to the constant speed c, resulting in transformations that are now part of history.

To honor one of the founders of our physical knowledge, in these papers we shall refer to *the Lorentz problem* the achievement of the invariance of locally varying speeds of light with an arbitrary functional dependence of the index of refraction.

During the century following Lorentz studies, the invariance of locally varying speeds of light was forgotten due to the reduction of light to photons propagating in vacuum irrespective of whether in exterior or interior conditions, with consequential use of Lorentz's invariance for the constant speed c.

Via rigorously proved *No Reduction Theorems* indicated in Paper I, Santilli established the impossibility of a consistent reduction of interior to exterior conditions thus bringing back to life the Lorentz's problem as a beautiful problem *per se,* in view of its practical value irrespective of whether light is reducible or not to photons moving in vacuum, as well as for photons themselves.

The various conceptual, mathematical, theoretical and experimental needs to re-examine the scattering theory presented in Paper I, render Lorentz's problem one of the most important problems in contemporary applied mathematics, whose solution can stimulate momentous advances in all quantitative sciences.

1.3. Insufficiencies of Lie's Theory. Santilli has dedicated his research life to the study of the Lorentz problem. The first contribution, as part of his Ph. D. Thesis in the mid 1960s, was to show that Lorentz's inability to achieve the desired invariance originated from insufficiencies of the background theory, *Lie's theory.* In fact, the applicability of said theory is notoriously restricted to linear, local and canonical systems at the classical level or unitary systems at the operator counterpart (where, in these papers, linearity is referred to the wavefunctuion, locality is referred to a finite set of isolated points, and canonicity or unitarity are referred to the respective time evolutions).

By contrast, the transition from the Minkowski metric characterizing the constant speed c to the deformed metric characterizing variable speed (1.1)

$$\eta = Diag.(1,1,1,-c^2) \rightarrow$$
$$\rightarrow \hat{\eta} = Diag.(1,1,1,-\frac{c^2}{n^2(x,v,\xi,\omega,\psi,\partial\psi,...)}), \tag{1.2}$$

has been shown in Paper I as characterizing systems that are generally *nonlinear in the wavefuynctsion, nonlocal of integral character, and non-canonical or nonunitary in their time evolution,* It is then evident that Lie's theory. while so effictive for the constant speed c, is generally *inapplicable* for the case of local speeds (1.1) (and certainly not "violated" because not conceived for the systems considered).

1.4. Lie-Admissible Covering of Lie's Theory. As part of his Ph. D. thesis, in order to broaden the representational capabilities of Lie's theory, Santilli proposed in 1967 [3] the first known *deformation of Lie algebras* in the physics literature with product

$$(A,B) = p \times A \times B - q \times B \times A, \tag{1.3}$$

where $p, q, p \pm q$ are non-null scalars (denoted λ and μ in Ref. [3]), A, B are matrices of the same dimension, and $A \times B$ is the conventional associative product according to the notations set forth in Paper I. Santilli called deformations (1.3) *mutations of Lie algebras* due to the evident loss of Lie's axioms, and proved that they characterize *Lie-admissible and Jordan-admissible algebras* according to Albert (in the sense that their attached antisymmetry and symmetric algebras are Lie and Jordan, respectively).

The proposal was intended to characterize the following *Lie-admissible generalization of Heisenberg's equations* for the dynamical evolution of a Hermitean operator A in the following infinitesimal and finite forms

$$i \times \frac{dA}{dt} = (A,H) = p \times A \times H - q \times H \times A, \tag{1.4a}$$

$$A(t) = e^{H\times q\times t\times i} \times A(0) \times e^{-i\times t\times q\times H}, \tag{1.4b}$$

By recalling that Lie algebras characterize closed-conservative systems reversible over time, proposal [3] essentially recommended the construction of a *Lie-admissible covering of Lie's theory* for the characterization of open, nonconservative and irreversible systems evidently in view of the non-null *time rate of variations of the energy* $i \times dH/dt = (H, H) \neq 0$.

In 1978, Santilli [4] proposed the *most general possible Lie-admissible and Jordan-admissible deformations-mutations of Lie algebras* with product

$$(A\hat{,}B) = A \times R \times B - B \times S \times A = A < B - B > A, \tag{1.5}$$

where $R, S, R \pm S$ are now fixed nonsingular operators with an arbitrary, nonlinear and nonlocal functional dependence on any needed quantity (including the wavefunction and its derivatives), which brackets resulted to characterize the *most general possible algebra* as known in mathematics (characterized by a bilinear composition law verifying the right and left distributive and scalar laws). Therefore, algebras with product (1.5) contain as particular cases associative, Lie, Jordan, supersymmetric, flexible and any otter possible algebra.

Ref. [4] then presented the initiation of a joint *Lie-admissible and Jordan-admissible covering o Lie's theory* in its various branches, including the lifting of the universal enveloping algebra with generalized Poincaré-Birkhoff-Witt theorem, Lie algebras, Lie's (transformation) groups and the representation theory.

Product (1.5) was obtained by using the most general possible *nonunitary* transformation of product (1.3), and was suggested as the foundations of the following *Lie-admissible and Jordan-admissible deformations-mutations of Heisenberg's equations* proposed in the joint paper [5] with infinitesimal and finite forms

$$i \times \frac{dA}{dt} = (A\hat{,}H) = A \times R \times H - H \times S \times A = A < H - H > A, \tag{1.6a}$$

$$A(t) = e^{H\times S\times t\times i} \times A(0) \times e^{-i\times t\times R\times H}, \tag{1.6b}$$

$$R = S^{\dagger} \tag{1.6c}$$

The above equations were proposed as the foundations of *hadronic mechanics* for the representation of the most general possible open, nonconservative, irreversible and single-valued systems with potential interactions represented by the nonconserved Hamiltonian H, and contact nonpotential, nonlinear, nonlocal-integral and nonunitary interactions represented by the operators R, S.

Generalized dynamical equations (1.6) were originally formulated on conventional Hilbert spaces over conventional fields. Subsequent studies indicated that the equations verified the *Theorems of Catastrophic Mathematical and Physical Inconsistencies of Noncanonical and Nonunitary Theories* (see Refs. [6-12] of Paper I) because not preserving over time the basic units of measurements, the observability of physical quantities, the numerical predictions, etc.

The resolution of the above inconsistencies required decades of additional research. The first major advance occurred in 1993 with the discovery of the *genonumbers* and *genofields* [6], namely, fields with a fixed order of of *all* multiplications to the right (representing motion forward in time) and an arbitrary right and left generalized unit called *genounit for the ordering to the right,*

$$n > m = n \times S \times m, \ \hat{I}^{>} = S^{-1}, \tag{1.7}$$

with the corresponding ordering of *all* multiplications to the left (representing motion backward in time) with related *genounit for the ordering to the left*

$$n < m = n \times R \times m, \ {}^{<}\hat{I} = R^{-1}, \tag{1.8}$$

where the word "genotopy" [4] was used in the Greek meaning of inducing new axioms.

In turn, the above genofields stimulated corresponding *two* genotopies, one to the right and, separately, one to the left, of functional analysis, metric spaces, geometries, enveloping associative algebras, etc. Despite all these efforts, the resolution of the inconsistency theorems remained elusive for years.

A breakthrough occurred in the mathematical memoir [7] of 1996 with the discovery of the new *genodifferential calculus to the right or to the left.* The first invariance over time of deformations-mutations of Lie algebras was proved in paper [8] of 1997. Final maturity in the axiomatic structure of Lie-admissible formulations was achieved in memoir [9] of 2006 that also presented the first known connection between mechanics and thermodynamics, by showing that the irreversibility of thermodynamical laws originates at the ultimate level of nature, in full confirmation of the No Reduction Theorems indicated earlier.

Readers should be aware that, in view of their only known axiomatically consistent characterization of irreversible processes (thus including energy releasing processes) in a way directly compatible with thermodynamics, Lie-admissible formulations have been the subject of rather vast studies since the time of Santilli's original proposal of 1978 [4], including mathematical, physical, chemical as well as industrial research (see monographs [10-27], references quoted therein and general bib biography in Volume [16a]).

We should mention that, twenty years following the origination of the parametric deformations [3] and ten years following the proposal of the operator deformations [4,5] (with related rather vast literature of the time including four monographs [10,11], five workshops on Lie-admissibility and an international conference [16a]), there was the appearance of a very large number of papers on parametric deformations of Lie algebras with the simpler product $A \times B - q \times B \times A$, generally without the quotation of their origination [3-5], as well as generally without the identification of their joint Lie-admissible and Jordan admissible character, despite their historical and technical values.

It is important for these papers to indicate that all the latter deformations have been proved to verify the theorems of catastrophic inconsistencies when formulated on conventional spaces over conventional fields (see Refs. [6-12] of Paper I). The words "deformations-isotopies" of the titles of the various sections of this paper stand to indicate that their field in applied mathematics is that nowadays vastly referred to as "deformations," although identically reformulated as "isotopies" to resolve said inconsistencies.

1.5. Lie-Isotopic Covering of Lie's Theory. These papers are intended for concrete applications to the elaboration of scattering data. As such, if initially presented with excessive mathematical complexities (as needed for the consistent treatment of irreversible scattering processes), these papers could be beyond the reach of most phenomenolooists.

This is the reason that has suggested in Paper I the restri+tion of these initial studies to *reversible* scattering processes, and then the passage to the more complex irreversible events only subsequently. As an example, the restriction to reversible processes eliminates the need of the time ordering of all products, with consequential major simplification of the formalism.

Most importantly from the viewpoint of applied mathematics, *the restriction to reversible scattering processes permits the preservation of Lie's axioms, despite the admission of nonlinear, nonlocal and noncanonical or nonunitary effects.*

In fact, Santilli identified in the original proposal [4] of 1978, the following particularization of his Lie-admissible and Jordan-admissible product (1.5)

$$[A\hat{,}B] = A\hat{\times}B - B\hat{\times}A =$$

$$= A \times T(x, v, \xi, \omega, \psi, \partial\psi, ...) \times B - B \times T(x, v, \xi, \omega, \psi, \partial\psi, ...) \times A, \quad (1.9a)$$

$$R = S = T = T^{\dagger} > 0, \ \hat{I}(x, v, \xi, \omega, \psi, \partial\psi, ...) = 1/\hat{T}(x, v, \xi, \omega, \psi, \partial\psi, ...) > 0, \quad (1.9c)$$

where $\hat{I}(x, v, \xi, \omega, \psi, \partial\psi, ...)$ and $\hat{T}(x, v, \xi, \omega, \psi, \partial\psi, ...)$ are the *isounit* and the*isotopic element* at the foundation of the mathematics of Paper I, where one should note that quantities (1.9b) have the same functional dependence of local speed (1.1).

It is easy to verify that product (1.9) does indeed verify Lie's axioms. Consequently, the ensuing *deformations of Lie algebras* were called *isotopic* [4] by Santilli in their Greek meaning of preserving the original topology, a main characteristics that we have used in the very name of the isoscattering theory. In the same paper [4], Santilli then proposed a step by step isotopic generalization of Lie's theory that has remained structurally unchanged to this day (except for the subsequent reformulation on isospaces over isofields), and it is today known as the *Lie-Santilli isotheory* [18-27].

The main idea of said isotheory is that of preserving unchanged the generators of a given Lie symmetry and changing instead all their operations in an axiom-preserving way (as a condition to have an isotopy) [4]. The implementation of this idea require the lifting of the conventional associative product $A \times B$ into the axiom-reserving isoassociative form $A \times T \times B = A\hat{\times}B$ that, in turn, implies the lifting of the Lie product $[A.B]$ into the axiom-preserving form (1.9).

This seemingly elementary idea has important implications for the scattering theory. By recalling that the generator of a Lie symmetry represent conserved quantities, the preservation of the generators in the transition from the conventional to the isotopic scattering theory implies the preservation of all conventionally conserved quantities. However, the appearance of the isotopic element T in the product itself implies that said preservation occurs under nonlinear, nonlocal and noncanonical or nonunitary internal effects, thus warranting a reinspection of the data elaboration via the conventional linear, local and unitary scattering theory.

Since the covering isotheory is at the foundations o these papers, it appears recommendable to outline its main elements in a language accessible to phenomenologists and specialized to scattering problems, not only for notational scopes, but also to avoid possible insidious misrepresentations in the event of referral to a variety of seemingly different presentations existing in the literature. Also, most of the "objections" raised by colleagues in a shorter versions of these papers were essentially due to a lack of inspection of the Lie-Santilli isotheory in the disparate literature or, more insidiously, due to the inspection of presentations prior to the achievement of invariance. Also, some of the results on Lie-isotopic studies are at times presented in the broader Lie-admissible context, as it is often the case of the original proposal [4].

As it was the case for other isotopies outlined in Paper I, the Lie-Santilli isotheory coincides with the conventional Lie theory at the abstract, realization-free level by conception and construction to such an extent that they can be presented at the pure mathematical level with the same symbols subjected to different realizations. However, such an abstract presentation would render quite difficult the practical applications of the isoscattering theory. Consequently, we shall outline below the specialization of the isotheory with emphasis on its applied version, namely, in its *projection* on conventional spaces over conventional fields.

Additionally, the reader should be aware that the original presentations verified the inconsistency theorems due to lack of invariance over time. In fact, the Lie-Santilli theory reached maturity only following the discovery of the *isonumbers* in 1993 [6] and of the *isodifferential calculus* in 1996 [7], discoveries that followed the otherwise excellent presentation by Tsagas and Sourlas [20] of 1993. Therefore, the outline below is based on all structural elements of the original proposal [4] formulated on isospaces over isofields [6] and via the isodifferential calculus [7].

UNIVERSAL ENVELOPING ISOASSOCIATIVE ALGEBRAS
Let $E = E(L)$ be the *universal enveloping associative algebra* of an N-dimensional Lie algebra L with ordered (Hermitean) generators $X_k,\ k = 1, 2, \ldots, N$, and attached antisymmetric algebra isomorphic to the Lie algebra, $[E(L)]^- \approx L$ over a field F (of characteristic zero), and let the infinite-dimensional basis $I, X_k,\ X_i \times X_j,\ i \leq j, \ldots$ of $E(L)$ be characterized by the *Poincaré-Birkhoff-Witt theorem.* We then have the following

THEOREM 1.5.1 [4]: (Poincaré-Birkhoff-Witt-Santilli theorem): The isocosets of the isounit and of the standard isomonomials

$$\hat{I},\ \ X_k,\ \ \hat{X}_i \hat{\times} \hat{X}_j,\ i \leq j,\ \ \hat{X}_i \hat{\times} \hat{X}_j \hat{\times} \hat{X}_k,\ i \leq j \leq k,\ \ldots, \tag{1.10}$$

form an (infinite dimensional) basis of the universal enveloping isoassociative algebra $\hat{E}(\hat{L})$ (also called isoenvelope for short) of a Lie-Santilli isoalgebra $\hat{L}$.

The first application of the above theorem, also formulated in Ref. [4] and then reexamined by various authors, is a rigorous characterization of the *isoexponentiation,* Eq. (3.4) of Paper I, i.e.,

$$\hat{e}^{\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}} =$$

$$= \hat{I} + \hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X} \hat{/} \hat{1}! + (\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}) \hat{\times} (\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}) \hat{/} \hat{2}! + \ldots =$$

$$= \hat{I} \times (e^{i \times w \times T \times X}) = (e^{i \times w \times X \times T}) \times \hat{I}, \tag{1.11a}$$

$$\hat{i} = i \times \hat{I}, \hat{w} = w \times \hat{I} \in \hat{F}. \tag{1.11b}$$

where we continue to use the notation of Paper I according to which quantities with a "hat" are formulated on isospaces over isofields and those without are formulated on conventional spaces over conventional fields.

The nontriviality of the Lie-Santilli isotheory is illustrated by the emergence of the nonlinear, nonlocal and noncanonical or nonunitary isotopic element T directly in *the exponent,* thus ensuring the desired generalization.

LIE-SANTILLI ISOALGEBRAS.
As it is well known, Lie algebras are the antisymmetric algebras $L \approx [\xi(L)]^-$ attached to the universal enveloping algebras $\xi(L)$. This main characteristic is preserved although enlarged under isotopies as expressed by the following

THEOREM 1.5.2 [4] (Lie-Santilli Second theorem): The antisymmetric isoalgebras $\hat{L}$ attached to the isoenveloping algebras $\hat{E}(\hat{L})$ verify the isocommutation rules

$$[\hat{X}_i \hat{,} \hat{X}_j] = \hat{X}_i \hat{\times} \hat{X}_j - \hat{X}_j \hat{\times} \hat{X}_i =$$

$$= X_i \times T(x, v, \xi, \omega, \psi, \partial\psi, ...) \times X_j - X_j \times T(x, v, \xi, \omega, \psi, \partial\psi, ...) \times X_i =$$

$$= \hat{C}_{ij}^k(x, v, \xi, \omega, \psi, \partial\psi, ...) \hat{\times} \hat{X}_k = C_{ij}^k(x, v, \xi, \omega, \psi, \partial\psi, ...) \times X_k, , \quad (1.12)$$

where the C's, called the "structure isofunctions" of $\hat{L}$, generally have an explicit dependence on local variables, and are restricted by the conditions (Lie-Santilli Third Theorem)

$$[X_i \hat{,} X_j] + [X_j \hat{,} X_i] = 0, \quad (1.13a)$$

$$[[X_i \hat{,} X_j] \hat{,} X_k] + [[X_j hat, X_k] \hat{,} X_i] + [[X_k hat, X_i] \hat{,} X_j] = 0. \quad (1.13b)$$

It was stated in the original proposal [4] that *all isoalgebras $\hat{L}$ are isomorphic to the original algebra L for all positive-definite isotopic elements.* In other words, the isotopies *cannot* characterize any new Lie algebras algebra because all possible Lie algebras are known from Cartan classification. Therefore, Lie-Santilli isoalgebras merely provide *new nonlinear, nonlocal and noncanonical or nonunitary realizations of existing Lie algebras.*

LIE-SANTILLI ISOGROUPS.
Under certain integrability and smoothness conditions hereon assumed, Lie algebras L can be "exponentiated" to their corresponding Lie transformation groups G and, vice-versa, Lie transformation groups G admit corresponding Lie algebras L when computed in the neighborhood of the unit I.

These basic properties are preserved under isotopies although broadened to the most general possible, axiom-preserving nonlinear, nonlocal and non-canonical transformations groups according to the following:

THEOREM 1.5.3 [4] (Lie-Santilli iaogroups): The isogroup characterized by finite (integrated) form $\hat{G}$ of isocommutation rules (1.12) on an isospace $\hat{S}(\hat{x}, \hat{F})$ over an isofield $\hat{F}$ with common isounit $\hat{I} = 1/\hat{T} > 0$ is a group mapping each element $\hat{x} \in \hat{S}$ into a new element $\hat{x}' \in \hat{S}$ via the isotransformations

$$\hat{x}' = \hat{g}(\hat{w})\hat{\times}\hat{x}, \;\; \hat{x}, \hat{x}' \in \hat{S}, \;\; \hat{w} \in \hat{F}, \tag{1.14}$$

with the following isomodular action to the right:
1) The map $\hat{g}\hat{\times}\hat{S}$ into $\hat{S}$ is isodifferentiable $\forall \hat{g} \in \hat{G}$;
2) $\hat{I}$ is the left and right unit

$$\hat{I}\hat{\times}\hat{g} = \hat{g}\hat{\times}\hat{I} \equiv \hat{g}, \;\; \forall \hat{g} \in \hat{G}; \tag{1.15}$$

3) the isomodular action is isoassociative, i.e.,

$$\hat{g}_1\hat{\times}(\hat{g}_2\hat{\times}\hat{x}) = (\hat{g}_1\hat{\times}\hat{g}_2)\hat{\times}\hat{x}, \;\; \forall \hat{g}_1, \hat{g}_2 \in \hat{G}; \tag{1.16}$$

4) in correspondence with every element $\hat{g}(\hat{w}) \in \hat{G}$ there is the inverse element $\hat{g}^{-\hat{I}} = \hat{g}(-\hat{w})$ such that

$$\hat{g}(\hat{0}) = \hat{g}(\hat{w})\hat{\times}\hat{g}(-\hat{w}) = \hat{I}; \tag{1.17}$$

5) the following composition laws are verified

$$\hat{g}(\hat{w})\hat{\times}\hat{g}(\hat{w}') = \hat{g}(\hat{w}')\hat{\times}\hat{g}(\hat{w}) = \hat{g}(\hat{w} + \hat{w}'), \forall \hat{g} \in \hat{G}, \;\; \hat{w} \in \hat{F}; \tag{1.18}$$

with corresponding isomodular action to the left, and general expression

$$\hat{g}(\hat{w}) = \prod_k \hat{e}^{\hat{i}\hat{\times}\hat{w}_k\hat{X}_k}\hat{\times}\hat{g}(0)\hat{\times}\prod_k \hat{e}^{=\hat{i}\hat{\times}\hat{w}_k\hat{X}_k}, \tag{1.19}$$

Another important property is that conventional group composition laws admit a consistent isotopic lifting, resulting in the following

THEOREM 1.5.4 [4] (Baker-Campbell-Hausdorff-Santilli theorem):

$$(\hat{e}^{\hat{X}_1})\hat{\times}(\hat{e}^{\hat{X}_2}) = \hat{e}^{\hat{X}_3}, \tag{1.20a}$$

$$\hat{X}_3 = \hat{X}_1 + \hat{X}_2 + [\hat{X}_1 \hat{,} \hat{X}_2]\hat{/}\hat{2} + [(\hat{X}_1 - \hat{X}_2)\hat{,}[\hat{X}_1 \hat{,} \hat{X}_2]]\hat{/}\hat{12} + \ldots\ . \tag{1.20b}$$

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits $\hat{I}_1$ and $\hat{I}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\times} \hat{G}_2$ is the isogroup of all ordered pairs

$$(\hat{g}_1, \hat{g}_2), \quad \hat{g}_1 \in \hat{G}_1, \hat{g}_2 \in \hat{G}_2, \tag{1.21}$$

with isomultiplication

$$(\hat{g}_1, \hat{g}_2)\hat{\times}(\hat{g}_1', \hat{g}_2') = (\hat{g}_1 \hat{\times} \hat{g}_1', \hat{g}_2 \hat{\times} \hat{g}_2'), \tag{1.22}$$

total isounit $(\hat{I}_1, \hat{I}_2)$ and inverse $(\hat{g}_1^{-\hat{I}_1}, \hat{g}_2^{-\hat{I}_2})$.

The following particular case is important for the isotopies of inhomogeneous groups. Let $\hat{G}$ be an isogroup with isounit $\hat{I}$ and $\hat{G}_{\hat{a}}$ the group of all its inner automorphisms. Let $\hat{G}^o_{\hat{a}}$ be a subgroup of $\hat{G}_{\hat{a}}$ with isounit $\hat{I}^o$, and let $\Lambda(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}_{\hat{a}}$. The *semidirect isoproduct* $\hat{G} \hat{\times} \hat{G}^o_{\hat{a}}$ is the isogroup of all ordered pairs $(\hat{g}, \hat{\Lambda})\hat{\times}(\hat{g}^o, \hat{\Lambda}^o)$ with total isounit

$$I_{tot} = \hat{I} \times \hat{I}^o. \tag{1.23}$$

The studies of the isotopies of the remaining aspects of the structure of Lie groups is then consequential. It is hoped the reader can see from the above elements that the entire conventional Lie theory does indeed admit a consistent and nontrivial lifting into the covering Lie-Santilli formulation.

Among a considerable number of mathematical papers on the Lie-Santilli isotheory listed in the Comprehensive Bibliography of Volume [16a], we quote in particular the readable review by J. V. Kadeisvili [28], an excellent presentation of the all fundamental isotopology by R. M. Falcon Ganfornina and J. Nunez Valdes [29], and the unification of all simple Lie algebras of a given dimension (excluding exceptional algebras) into one single Santilli isotope of the same dimension by Gr. T. Tsagas [30] (see also the review of the latter unification in Volume [16c]).

1.6. The Fundamental Theorem for Isosymmetries. The fundamental symmetries of the 20-th century physics characterize point-like abstractions of particles in vacuum under linear, local and potential interactions, and are given by the *Galilei symmetry* $G(3.1)$ for nonrelativistic treatment, the *Lorentz-Poincaré symmetry* for relativistic formulations, the $SU(3)$ symmetry for particle classifications, the gauge symmetry, and others.

A central objective of hadronic mechanics is the broadening of these fundamental symmetries to represent extended, nonspherical and deformable particles under linear and nonlinear, local and nonlocal and potential as well as nonpotential interactions *in such a way to preserve the original symmetries at the abstract level.*

This central objective is achieved by the following property first proved by Santilli in Ref. [13b]¿

THEOREM 1.5,5: Let G be an N-dimensional Lie symmetry of a K-dimensional metric or pseudo-metric space $S(x, m, F)$ over a field F,

$$G: \quad x' = \Lambda(w) \times x, \quad y' = \Lambda(w) \times y, \quad x, y \in \hat{S}, \tag{1.24a}$$

$$(x' - y')^\dagger \times \Lambda^\dagger \times m \times \Lambda \times (x - y) \equiv (x - y)^\dagger \times m \times (x - y), \tag{1.24b}$$

$$\Lambda^\dagger(w) \times m \times \Lambda(w) \equiv m. \tag{1.24c}$$

Then, all infinitely possible isotopies $\hat{G}$ of G acting on the isospace $\hat{S}(\hat{x}, \hat{M}, \hat{F})$, $\hat{M} = \hat{m} \times \hat{I} = (\hat{T}_i^k \times m_{kj}) \times \hat{I}$ characterized by the same generators and parameters of G and the infinitely possible, common isounits $\hat{I} = 1/\hat{T} > 0$ leave invariant the isocomposition

$$\hat{G}: \quad x' = \hat{\Lambda}(w) \times x, \quad y' = \hat{\Lambda}(w) \times y, \quad x, y \in \hat{S}, \tag{1.25a}$$

$$(x' - y')^\dagger \times \hat{\Lambda}^\dagger \times \hat{m} \times \hat{\Lambda} \times (x - y) \equiv (x - y)^\dagger \times \hat{m} \times (x - y), \tag{1.25b}$$

$$\hat{\Lambda}^\dagger(\hat{w}) \times \hat{m} \times \hat{\Lambda}(\hat{w}) \equiv \hat{m}. \tag{1.25c}$$

and all infinitely possible so constructed isosymmetries $\hat{G}$ are locally isomorphic to the original symmetry G.

For a proof one may inspect Section 1.2 of Ref. [16b].

To achieve a technical understanding of the Lie-Santilli isotheory and of the isoscattering theory, the reader should note that, *while a given Lie* *symmetry G is unique as well known, there can be an infinite number of covering isosymmetries $\hat{G}$ with generally different explicit forms o the transformations due to the infinite number of possible isotopic elements.*

In fact, systems are characterized by the Hamiltonian H in the conventional scattering theory with trivial unit $I = Diag.(1, 1, ..., 1)$. In this case, changing the Hamiltonian implies the referral to a different system, but the symmetry transformations remain the same. In the isoscattering theory, systems are characterized by the Hamiltonian H *plus the isotopic element* T. In this case, changing the isotopic element implies the referral to a different systems as well as the characterization of generally different transformations due to the appearance of the isotopic element in the very structure of the isosymmetry.

Note also that all possible isosymmetries can be explicitly and uniquely constructed via the sole knowledge of the conventional symmetry and the isotopic element. in fact, as implied by Theorem 1.5.5, the existence of the original symmetry plus the condition $\hat{I} > 0$ ensure verification of the integrability conditions for the existence of finite transformations, a property hereon tacitly implied.

1.7. Simple Construction of the Lie=Santilli Isotheory. A simple method has been identified in Refs. [13,16] for the construction of the Lie-Santilli isotheory, all its underlying isomathematics and all physical methods to be studied in the these papers. This method is important because it permits a simple implementation of scattering models into their isotopic form. The method consists in:

(i) Representing all conventional interactions with a Hamiltonian H and all non-Hamiltonian interactions and effects with the isounit $\hat{I}$;

(ii) Identifying the latter interactions with a nonunitary transform

$$U \times U^{\dagger} = \hat{I} \neq I \tag{1.26}$$

and

(iii) Subjecting the *totality* of conventional mathematical and physical quantities and all their operations to the above nonunitary transform, resulting in expressions of the type

$$I \to \hat{I} = U \times I \times U^\dagger = 1/\hat{T}, \tag{1.27a}$$

$$a \to \hat{a} = U \times a \times U^\dagger = a \times U \times U^\dagger = a \times \hat{I},\ a \in F, \tag{1.27b}$$

$$e^A \to U \times e^A \times U^\dagger = \hat{I} \times e^{\hat{T} \times \hat{A}} = (e^{\hat{A} \times \hat{T}}) \times \hat{I}, \tag{1.27d}$$

$$A \times B \to U \times (A \times B) \times U^\dagger =$$

$$= (U \times A \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times B \times U^\dagger) = \hat{A} \hat{\times} \hat{B}, \tag{1.27c}$$

$$[X_i, X_j] \to U \times [X_i X_j] \times U^\dagger =$$

$$= [\hat{X}_i \hat{,} \hat{X}_j] = U \times (C^k_{ij} \times X_k) \times U^\dagger = \hat{C}^k_{ij} \hat{\times} \hat{X}_k =$$

$$= C^k_{ij} \times \hat{X}_k, \tag{1.27e}$$

$$< \psi| \times |\psi > \to U \times < \psi| \times |\psi > \times U^\dagger =$$

$$=< \psi| \times U^\dagger \times (U \times U^\dagger)^{-1} \times U \times |\psi > \times (U \times U^\dagger) =$$

$$=< \hat{\psi}|\hat{\times}|\hat{\psi} > \times \hat{I}, \tag{1.27f}$$

$$H \times |\psi > \to U \times (H \times |\psi >) = (U \times H \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times |\psi >) =$$

$$= \hat{H} \hat{\times} |\hat{\psi} >, etc. \tag{1.27g}$$

The above simple rules permit the explicit construction of all needed *regular isotopies* as defined and illustrated ui Section 1.9 (eigenvalkue preserving mapps), including: algebras, groups, symmetries, eigenvalues equations and all needed aspects [13]. It should be stressed that the above method is not applicable for the *irreguilar isotopies* as also defined and illustrated in Section 1.9 (eigenvalue mutating images) for which no map ois known at this writing.

Note finally that *serious inconsistencies emerge in the event even one single quantity or operation is not subjected to the above nonunitary map.* In the absence of comprehensive liftings, we would have a situation equivalent to the elaboration of quantum spectral data of the hydrogen atom with isomathematics, resulting in dramatic deviations from reality.

1.8. Invariance of the Lie-Santilli Isotheory. It is easy to see that the application of an additional nonunitary transform

$$W \times W^\dagger \neq I, \tag{1.28}$$

to expressions (1.27) causes the *lack of invariance,* with consequential activation of the catastrophic inconsistency theorems reviewed in Paper I, such as the *change of the basic isounit*

$$\hat{I} \rightarrow \hat{I}' = W \times \hat{I} \times W^\dagger \neq \hat{I}, \qquad (1.29)'$$

that implies the loss of the represented system, let alone the lack of invariance of a physical theory over time, or the lack of invariance of an isosymmetry under its own action,

However, as indicated in Paper I, any given nonunitary transform can be identically rewritten in the isounitary form,

$$W \times W^\dagger = \hat{I}, \quad W = \hat{W} \times \hat{T}^{1/2}, \qquad (1.30a$$

$$W \times W^\dagger = \hat{W} \hat{\times} \hat{W}^\dagger = \hat{W}^\dagger \hat{\times} \hat{W} = \hat{I}, \qquad (1.30b)$$

under which we have the invariance of the isounit and isoproduct [7]

$$\hat{I} \rightarrow \hat{I}' = \hat{W} \hat{\times} \hat{I} \hat{\times} \hat{W}^\dagger = \hat{I}, \qquad (1.31a)$$

$$\begin{gathered} \hat{A} \hat{\times} \hat{B} \rightarrow \hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{W}^\dagger = \\ = (\hat{W} \times \hat{T} \times \hat{A} \times \hat{T} \times \hat{W}^\dagger) \times (\hat{T} \times \hat{W}^\dagger)^{-1} \times \hat{T} \times (\hat{W} \times \\ \times \hat{T})^{-1} \times (\hat{W} \times \hat{T} \times \hat{B} \times \hat{T} \times \hat{W}^\dagger) = \\ = \hat{A}' \times (\hat{W}^\dagger \times \hat{T} \times \hat{W})^{-1} \times \hat{B}' = \hat{A}' \times \hat{T} \times \hat{B}' = \hat{A}' \hat{\times} \hat{B}', \ etc. \end{gathered} \qquad (1.31b)$$

from which the invariance of the entire isotopic formalism follows.

Note that the invariance is ensured by the *numerically invariant values of the isounit and of the isotopic element under nonunitary-isounitary transforms,*

$$\hat{I} \rightarrow \hat{I}' \equiv \hat{I}, \qquad (1.32a)$$

$$A \hat{\times} B \rightarrow A' \hat{\times}' B' \equiv A' \hat{\times} B', \qquad (1.32b)$$

in a way fully equivalent to the invariance of Lie's theory and quantum mechanics, as expected to be necessarily the case due to the preservation of the abstract axioms under isotopies. The resolution of the inconsistencies for noninvariant theories is then consequential (see Paper I for details).

1.9. Regular and Irregular Pauli-Santilli Isomatrices. Due to the abstract identify of Lie and Lie-Santilli theories, as well as the simplicity of their interconnecting map of Section 1.7, it is at times believed that the isotopies are trivial. The best way to dispel this erroneous perception is via the isorepresentation theory for one of the central physical notions, that of spin.

Even though the isorepresentation theory is still vastly unexplored, the studies conducted until now have been sufficient to identify the existence of two classes, the *regular isorepresentations,* occurring under the preservation of the original structure constants, and the *irregular isorepresentations,* occurring under the alteration of the original structure constants.

The basic symmetries of the 20th century particle physics have been those of the *rotational symmetry* $SO(3)$ and the *spin symmetry* $SU(2)$. The corresponding isosymmetries $\hat{SO}(3)$ were studied by Santilli in the original proposal [5] of 1978 as well as in the two subsequent papers [31,32] of 1985. Isosymmetries $\hat{SU}(2)$ were first studied also by Santilli in paper [33] of 1993 and [34] of 1998 with the following main results:

CASE I: REGULAR PAULI-SANTILLI ISOMATRICES.
This is the case that, by definition, implies the *preservation of the conventional spin* $1/2$, although with new degrees of freedom nonexistent in the conventional notion of spin. The related regular two-dimensional irreducible isorepresentation of $\hat{SU}(2)$ are today known as *regular Pauli-Santilli isomatrices.*

This first notion of *hadronic spin,* that is, spin characterized by hadronic mechanics, is assumed for *low energy reversible scattering processes.* The assumption essentially implies that, as an example, an electron maintains its spin $1/2$ in the transition from motion in vacuum to motion within the scattering region, although in a generalized way identified below. As we shall see in Section 3, this assumption implies the preservation within the scattering region of the Fermi-Dirac statistics and Pauli's exclusion principle.

By remembering the lack of uniqueness of the isounits and related isotopic element, the simplest regular two-dimensional irreducible isorepresentations of $\hat{SU}(2)$ are characterized by the lifting of the two-dimensional complex-valued unitary space with metric $\delta = Diag.(1.1)$ into the isotopic image [33,34]

$$\hat{I} = Diag.((n_1^2, n_2^2), \ \hat{T} = Diag.(1/n_1^2, 1/n_2^2), \tag{1.33a}$$

$$\hat{\delta} = \hat{T} \times \delta = Diag.(1/n_1^2, 1/n_2^2), \tag{1.33b}$$

$$Det \ \hat{\delta} = (n_1 \times n_2)^{-2} = 1, \tag{1.33c}$$

with corresponding isounit and isotopic element

$$U \ timesU^\dagger = \hat{I} = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \ T = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix}. \tag{1.34}$$

The related lifting of Pauli's matrices can then be easily constructed via the methods of Section 1.7 as follows

$$\sigma_k \ \rightarrow \ \hat{\sigma}_k = U \times \sigma_k \times U^\dagger, \tag{1.35a}$$

$$U = \begin{pmatrix} i \times n_1 & 0 \\ 0 & i \times n_2 \end{pmatrix}, U^\dagger = \begin{pmatrix} -i \times n_1 & 0 \\ 0 & -i \times n_2 \end{pmatrix}, \tag{1.35b}$$

where the n's are well behaved nowhere null functions, resulting in the *regular Pauli-Santilli isomatrices* [*loc. cit*]

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1^2 \\ n_2^2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \times n_1^2 \\ i \times n_2^2 & 0 \end{pmatrix}, \ \hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}. \tag{1.36}$$

Another realization is given by *nondiagonal* nonunitary transforms [*loc. cit.*],

$$\begin{gathered} U = \begin{pmatrix} 0 & n_1 \\ n_2 & 0 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix}, \\ \hat{I} = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix}, \end{gathered} \tag{1.37}$$

with corresponding alternative version of the regular Pauli-Santilli isomatrices,

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1 \times n_2 \\ n_1 \times n_2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \times n_1 \times n_2 \\ i \times n_1 \times n_2 & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \tag{1.38}$$

or by more general realizations with Hermitean nondiagonal isounits $\hat{I}$ [15b].

All Pauli-Santilli isomatrices of the above regular class verify the following *isocommutation rules and isoeigenvalue equations* on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = \hat{\sigma}_i \times \hat{T} \times \hat{\sigma}_j - \hat{\sigma}_j \times \hat{T} \times \hat{\sigma}_i = 2 \times i \times \varepsilon_{ijk} \times \hat{\sigma}_k, \tag{1.39a}$$

$$\hat{\sigma}^{\hat{2}} \hat{\times} |\hat{\psi}\rangle =$$

$$(\hat{\sigma}_1 \times T \times \hat{\sigma}_1 + \hat{\sigma}_2 \times T \times \hat{\sigma}_2 + \hat{\sigma}_3 \times T \times \hat{\sigma}_3) \times T \times |\hat{\psi}\rangle = 3 \times |\hat{\psi}\rangle, \tag{1.39b}$$

$$\hat{\sigma}_3 \hat{\times} |\hat{\psi}\rangle = \hat{\sigma}_3 \times T \times |\hat{\psi}\rangle = \pm 1 \times |\hat{\psi}\rangle, \tag{5.39c}$$

thus preserving conventional structure constants and eigenvalues for spin 1/2 under non-Hamiltonian/nonunitary interactions.

An interesting interpretation has been proposed in Ref. [31] for the case

$$n_1^2 = \lambda, \ \ n_2^2 = \lambda^{-1}, \tag{1.40}$$

according to which *the Pauli-Santilli isomatrices provide an explicit and concrete realization of a kind of hidden variables,* in the sense that the variable λ is indeed hidden in the axioms of the $SU(2)$ symmetry, with the understanding that we are not referring to the traditional interpretation of hidden variables, such as the historical one by Bohm. Note that this new degree of freedom is absent in the conventional Lie theory and can be solely identified via the Lie-Santilli isotheory.

Irrespective of the type of hydden variable we are here referring to, the Pauli-Santilli isomatrices with characteristic quantity (1.40) have caused a reinspection of Bell's inequalities, local realism and all that due to the strictly unitary structure of the latter compared to the nonunitary character of the former. We regret being unable to outline these intriguing new vistas, and refer the interested reader to paper [34].

CASE II: IRREGULAR PAULI-SANTILLI ISOMATRICES.
As it is well known by experts in quantum mechanics, action-at-a-distance, potential interactions leave invariant the *intrinsic* characteristics of particles, such as spin. By comparison, as well known by experts in hadronic mechanics, contact non-Hamiltonian interactions generally cause alterations, called mutations, of all *intrinsic* characteristics of particles, including spin.

According to the Lie-Santilli isotheory, the mutations for spin 1/2 are characterized by the *irregular two-dimensional irreducible representations* of $\hat{SU}(2)$ known as the *irregular Pauli-Santilli isomatrices* that, by definition, do not preserve the spin 1/2 and, consequently, cannot be constructed via nonunitary transformations of conventional representations.

This case is assumed for the representation of particles at high energy originally having spin 1/2 when penetrating within hyperdense hadronic media, whether thos existing in the core of stars or inside very high energy scattering region. The main argument is that the belief that an electron preserves its spin 1/2 when in the core of a star does not appear to be plausible on various grounds, such as the loss of conventional quantized states within hyperdense media, the impossibility under the same conditions to possess a conserved angular momentum, and other reasons [13,16].

One illustrative example of irregular Pauli-Santilli isomatrices is given by [33.34]

$$\tilde{\sigma}_1 = \begin{pmatrix} 0 & n_1^2 \\ n_2^2 & 0 \end{pmatrix}, \ \ \tilde{\sigma}_2 = \begin{pmatrix} 0 & -i \times n_1^2 \\ i \times n_2^2 & 0 \end{pmatrix}, \ \ \tilde{\sigma}_3 = \begin{pmatrix} w \times n_1^2 & 0 \\ 0 & w \times n_2^2 \end{pmatrix}. \tag{1.41}$$

where w is the *mutation parameter*, with isocommutation rules

$$[\tilde{\sigma}_1, \tilde{\sigma}_2] = i \times w^{-1} \times \tilde{\sigma}_3, \ \ [\tilde{\sigma}_2, \tilde{\sigma}_3] = i \times w \times \tilde{\sigma}_1, \ \ [\tilde{\sigma}_3, \tilde{\sigma}_2] = i \times w \times \tilde{\sigma}_1, \tag{1.42}$$

and isoeigenvalues

$$\tilde{\sigma}^{\hat{2}} \hat{\times} |\hat{\psi}\rangle =$$

$$(\tilde{\sigma}_1 \times T \times \tilde{\sigma}_1 + \tilde{\sigma}_2 \times T \times \tilde{\sigma}_2 + \tilde{\sigma}_3 \times T \times \tilde{\sigma}_3) \times T \times |\hat{\psi}\rangle = (2 + w^2) \times |\hat{\psi}\rangle, \tag{1.43a}$$

$$\tilde{\sigma}_3 \hat{\times} |\hat{\psi}\rangle = \tilde{\sigma}_3 \times T \times |\hat{\psi}\rangle = \pm w \times |\hat{\psi}\rangle, \ \ w \neq 1, \tag{1.43b}$$

Additional examples of irregular Pauli-Santilli isomatrices can be found in Refs. [13,16].

The assumption of a mutated spin in hyperdense interior conditions evidently implies the inapplicability (rather than the violation) of the Fermi-Dirac statistics, Pauli's exclusion principle and other quantum mechanical laws, with the understanding that, by central assumption of Paper I, the scattering region as a whole must have conventional total quantum values because inspected from exterior conditions. Therefore, we are here referring to possible internal exchanges of angular momentum always in such a way to cancel out and yield total conventional values,

It should be indicated that we are here stressing the need to establish our knowledge in interior conditions via experiments rather than unverified assumptions, for which reason the isoscattering theory is proposed in the first place. The need to *test Pauli's exclusion principle under "external" strong interactions* was stresses since the title of paper [5] of 1978 and, after some 32 years, that call remains more valid than ever.

2.Deformations-Isotopies of Special Relativity.

2.1. Introduction. Following decades of research on the deformations-isotopies of Lie's theory, Santilli was finally in a position to construct the deformations-isotopies of all main aspects of the conventional *Lorentz-Poincaré (LP) symmetry,* including the isotopies of: the rotational symmetry [4,31,32]; the $SU(2)$-spin symmetry [33,34]; the Lorentz symmetry at the classical [35] and operator [36] levels; the Poincaré symmetry [37]; the spinorial covering of the Poincaré symmetry [38,39]; and the isotopies of the Minkowskian geometry 40]. The new symmetry is today known as the *Lorentz-Poincaré-Santilli isosymmetry,* or *LPS isosymmetry,* for short [18-27].

Following all the above preparatory research, Santilli was finally in a position to study the deformations-isotopies of special relativity into a form providing the invariant (rather than covariant) characterization of interior dynamical problems at large, including the interior of the scattering region, the locally varying speed of light or photons (1.1).

The difficulties inherent in the realization of this objective were compounded by Santilli's specific intent of *honoring Albert Einstein via the* *preservation of his axioms for interior dynamical problems, and the mere presentation of broader realizations,* so as to avoid the abuse of Einstein's name via the application of his axioms under conditions never intended for and never directly tested.

The above objective was achieved thanks to the universal LPS isosymmetry, as well as its local isomorphism to the conventional LP symmetry, resulting in the axiom-preserving deformations-isotopies of special relativity first presented in Refs. [35,36] of 1983 at the classical and operator levels, respectively, and then studied in a variety of subsequent works (see monographs [12] of 1991 for the first systematic treatment and subsequent presentations in monographs [13] of 1995 and [16] of 2008 with literature quoted therein), resulting in a covering relativity today known as *Santilli isorelativity* [18-27].

It should be indicated that numerous "deformations" of the Minkowski space, the Lorentz-Poincaré symmetry and special relativity exist in the literature. However, to our best knowledge, all of them appeared a decade following the original proposal [35] by generally adopting the same symbols and main terminology, often without the quotation of the originating works [35]. Numerous other attempts at generalizing special relativity exist in the literature of the past century, although they do not possess a universal symmetry, thus lacking uniqueness in their derivation.

All these studies are noncanonical at the classical level and nonunitary at the operator level as an evidently necessary condition for novelty, and are formulated on conventional spaces over conventional fields. As such, all these studies directly verify the *Theorems of Catastrophic Inconsistencies of Noncanonical and Nonunitary Theories,* Refs. [6.-12] of Paper I.

We regret to be unable to review these studies to prevent an excessive length, as well as risk partial, thus discriminatory listings. Nevertheless, it is hope that interested colleagues may inspect preceding broadening of special relativity because it is the hope of all theories, including those here proposed, to contain at best a grain of truth, and comparative analyses of different approaches are always scientifically valuable.

The isoscattering theory is based on Santilli deformations-isotopies of the Minkowskian geometry, the Lorentz-Poincaré symmetry and special relativity because said isotopies:

1) are *directly universal,* that is, admitting as particular cases of all possible $(3+1)$-dimensional generalizations of the Minkowskian spacetime (universality) directly in the frame of the experimenter without any transformation to hypothetical reference frames (direct universality), and have been proved to include as particular cases all other possible deformations via different expansions in terms of different parameters and with different truncations [41-43], thus reducing a variety of possibilities to a primitive isosymmetry [41-43];

2) have resolved said inconsistency theorems, thus being consistently applicable to actual measurements [13,16]; and

3) have significant experimental verifications in classical physics, particle physics, nuclear physics, superconductivity, chemistry, biology, astrophysics and cosmology (see Ref. [16d], Chapter 5 of Ref. [27] and paper [47]).

As it was the case for the Lie-Santilli isotheory, the objections received by the authors on an earlier and shorter version of these papers on the deformations-isotopies of special relativity, were primarily due to the inspection of inappropriate literature or to inconsistent presentations because not formulated on isospaces over isofields. Consequently, it appears recommendable to review the foundational elements of the field specialized to the scattering problem.

2.2. Deformation-Isotopies of the Minkowski Spacetime. As it is well known, the carrier space of relativistic scattering theory is the familiar Minkowski space $M(x, \eta, \mathcal{R})$, where we assume in these papers $x = (x^\mu), \mu = 1, 2, 3, 4, x^k = r^k, k = 1, 2, 3, x^4 = t$, and $\eta = Diag.(1, 1, 1, -c^2)$. Such a space is crucially dependent on the assumed basis unit, that of the Lorentz symmetry $I = Diag.(1, 1, 1, 1)$.

As customary in *relativistic hadronic mechanics* [13,16], the Minkowski spacetime with related Lorentz-Poincaré symmetry and special relativity are assumed as being *exact for the conditions clearly indicated by Einstein, i.e., for point-like particles and electromagnetic waves propagating in vacuum conceived as empty space,* under which conditions we have the constancy of the speed of light c for all possible inertial systems. Therefore, special relativity is assumed as being exact everywhere in the *exterior* of the scattering region.

For the interior of the scattering region, as indicated in Section 2 of Paper I and studied in more details in Ref. [47], there are no possible inertial reference frames and we solely have the privileged frame at rest with the scattering region itself. Additionally, according to incontrovertible experimental evidence, the high energy scattering region is not an empty sphere with point-like particles in its interior, as requested by the mathematical structure of quantum mechanics. Instead, the scattering region is a hyperdense medium characterized by the mutual penetration of the wavepackets of scattering particles irrespective of whether their charge distribution is extended or point-like.

The above and other aspects imply that *in the interior of the scattering region the speed of light in general and that of photons in particular is assumed as being a local variable* $C = c/n(x, v, \xi, \omega, \psi, \partial\psi, ...)$ according to Eq. (1.1). Most importantly, photons cannot be assumed as propagating in vacuum when in the interior of the scattering region due to its hyperdense character, as indicated in Section 1.1.

The locally varying character of the speed of light is geometrically represented via the assumption that *physical media alter the geometry of spacetime.* This assumption is necessary for any geometric representation of the variation of the speed of light, $c \rightarrow C = c/n(x, v, \xi, \omega, \psi, \partial\psi, ...)$. Equivalently, we can say that no variation of the speed of light is possible without a corresponding alteration of spacetime.

Following decades of studies, Santilli [35.36] proposed in 1983 the representation of the alteration of spacetime via the (axiom-preserving) deformations-isotopies of the Minkowski spacetime, today called *Minkowski-Santilli isospacetimes,* or *isospacetimes* for short, generally indicated with the symbols $\hat{M}(\hat{x}, \hat{\eta}, \hat{\mathcal{R}})$, and characterized by:

A) The *isounit* and related *isotopic element* usually assumed (from their positive-definiteness) to have the diagonal form (see Ref. 13b] for off-diagonal realizations)

$$\hat{I} = Diag(1/b_1^2, 1/b_2^2, 1/b_3^2, 1/b_4^2) = Diag.(n_1^2, n_2^2, n_3^2, n_4^2), \tag{2.1a}$$

$$\hat{T} = Diag(b_1^2, b_2^2, b_3^2, b_4^2) = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2, 1/n_4^2), \tag{2.1b}$$

B) The *isometric*

$$\Xi = \hat{T} \times \hat{m} = (T \times \eta) \times \hat{I} = \hat{\eta} \times \hat{I} =$$

$$= [Diag(b_1^2, b_2, b_3^2, -c^2 \times b_4^2) \times \hat{I} = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2, -c^2/n_4^2) \times \hat{I}, \quad (2.2)$$

C) The *isoinvariant*

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{\Xi}_{\mu\nu} \hat{\times} \hat{x}^\nu = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} =$$

$$= x^1 \times b_1^2 \times x^1 + x^2 \times b_2^2 \times x^2 + x^3 \times b_3^2 \times x^3 - t \times c \times b_4^2 \times t \times c) =$$

$$= (\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - t^2 \times \frac{c^2}{n_4^2}) \times \hat{I}, \quad (2.3)$$

where one should note that $\hat{\Xi}$ is an isomatrix (because its elements are isonumbers), while $\hat{\eta}$ is an ordinary matrix.

The quantities $b_\mu = 1/n_\mu$ are called the *characteristic quantities* of the considered scattering region that can be averaged to constants, as we shall see in Paper III. A rather frequent erroneous perception is that the quantities $b_\mu = 1/n_\mu$ are arbitrary parameters, while in reality they represent *measurable quantities.*

In fact, the space isounits $\hat{I}_k^k = n_k^2$ characterize the actual *size* (thus being of the order of $10^{-13} cm$) and *shape* (say, a spheroid ellipsoid)) of the scattering region that are indeed measurable. By contrast, "parameters" can assume arbitrary values thus generally having no connection with the actual size and shape of the physical region considered. In the elaboration of the isoscattering theory, the space isounits $\hat{I}_k^k = n_k^2$ are normalized to the perfect sphere of radius $\hat{I}_k^k = 1 fm$.

Similarly, the time isounit $\hat{I}_4^4 = n_4^2$ is a direct representation of the index of refraction that, as such, cannot possibly be a "parameter." More specifically, the time isounit provides a geometrization of the *density* of the scattering region (defined as the ratio between its energy and volume). In the isoscattering theory, the time isounit is normalized to the value for the vacuum $\hat{I}_4^4 = n_4^2 = 1$ for which $C = c/n = c$.

Finally, the characteristic quantities allow a representation of the *inhomogeneity* of physical media, e.g., via a dependence of the isounits on the local coordinates), as well as of their *anisotropy* (e.g., via different values between the space and time components of the isounit). Therefore, *the isoscattering theory allows, for the first time, a direct representation (i.e., a representation via the isometric) of the size, shape, density, inhomogeneity and anisotropy of the scattering region.*

The distinction between "parameters" and "characteristic quantities" is best illustrated by the Bose-Einstein correlation. Its treatment via relativistic quantum mechanics requires *four* arbitrary parameter of unknown origin, called "chaoticity parameters," that are fitted from the experimental data. By contrast, the representation of the same experimental data via relativistic hadronic mechanics yields space characteristic quantities providing a numerical representation of the actual shape of the fireball (a very elongated ellipsoid), while the forth component provides a numerical, representation of the density of the fireball in a way consistent with other experiments (see Ref. [16d], Chapter 5 of Ref. [27] and papers quoted therein).

The *Minkowski-Santilli isogeometry* has been worked out in detail in Ref. [40]. Its main implication is that the resulting geometrization of the interior scattering region includes in a unified form *all possible geometries in* $(3+1)$-*dimensions,* thus including the Riemannian, Finslerian and other geometries that are merely differentiated by the assumed isounit, although all formulated via the abstract axioms of the *Minkowski* space. Such a broadening is necessary for any realistic representation of scattering regions, e.g., because interior dynamical problems generally require metrics with an explicit dependence on velocities and other variables, thus rendering the sole Riemannian description excessively restrictive.

2.3. Deformations-Isotopies of the Lorentz-Poincaré Symmetry. Following, and only following the prior construction of the deformations-isotopies of Lie's theory outlined in Section 1.5, Santilli constructed systematic, step-by-step deformations-isotopies of all spacetime symmetries [12,31-40], including the isotopies of the Galilei symmetry and of the Lorentz-Poincaré symmetry.

Evidently, we cannot possibly review here these studies in details. However, to render this presentation minimally selfsufficient, we outline the rudiments of the *regular Lorentz-Poincaré-Santilli (LPS) isosymmetry* $\hat{P}(3.1)$ [37] specialized to the scattering problem, and leave to the interested reader the study of the nonrelativistic-Galilean counterpart from monographs [12].

By using the second theorem, Eq. (1.12), the *regular LPS isoalgebra* is characterized by the conventional generators and the isocommutation rules

[37]

$$[J_{\mu\nu}\hat{,}J_{\alpha\beta}] = i \times (\hat{\eta}_{\nu\alpha} \times J_{\beta\mu} - \hat{\eta}_{\mu\alpha} \times J_{\beta\nu} - \hat{\eta}_{\nu\beta} \times J_{\alpha\mu} + \hat{\eta}_{\mu\beta} \times J_{\alpha\nu}), \quad (2,4a)$$

$$[J_{\mu\nu}\hat{,}P_{\alpha}] = i \times (\hat{\eta}_{\mu\alpha} \times P_{\nu} - \hat{\eta}_{\nu\alpha} \times P_{\mu}), \quad (2,4b)$$

$$[P_{\mu}\hat{,}P_{\nu}] = 0, \quad (2,4c)$$

The *iso-Casimir invariants* of $\hat{P}(3.1)$ are given by [37]

$$\hat{C}_1 = \hat{I}(x, ...), \quad (2.5a)$$

$$\hat{C}_2 = P^{\hat{2}} = P_{\mu} \hat{\times} P^{\mu} = P^{\mu} \times \hat{\eta}_{\mu\nu} \times P^{\nu} =$$

$$= P_k \times g_{kk} \times P_k - p_4 \times g_{44} \times P_4, \quad (2,5b)$$

$$\hat{C}_2 3 = W^{\hat{2}} = W_{\mu} \hat{\times} W^{\mu}, \;\; W_{\mu} = \hat{\epsilon}_{\mu\alpha\beta\rho} \hat{\times} J^{\alpha\beta} \hat{\times} P^{\rho}, \quad (2.5c)$$

and they are at the foundation of classical and operator *isorelativistic kinematics* [13,16].

Since $\hat{I} > 0$, it is easy to prove that *the LPS isosymmetry is isomorphic to the conventional symmetry.* It then follows that *the isotopies increase dramatically the arena of applicability of the Lorentz-Poincaré symmetry, from the sole Minkowskian spacetime to all infinitely possible isospacetimes* (2.3).

By using Theorem 1.5.3 on the isogroup, the main components of the *regular LPS isotransformations* can be presented as follows in their projection on conventional spacetime:

(1) **Regular isorotations** $\hat{SO}(3)$, first presented in Ref. [31,32], here expressed in the (1,2)-plane (see monograph [13b] for the general case)

$$x^{1'} = x^1 \times \cos[\theta \times (n_1 \times n_2)^{-1}] - x^2 \times \frac{n_1^2}{n_2^2} \times \times \sin[\theta \times (n_1 \times n_2)^{-1}], \quad (2.6a)$$

$$x^{2'} = x^1 \times \frac{n_2^2}{n_1^2} \times \sin[\theta \times (n_1 \times n_2)^{-1}] + x^2 \times \cos[\theta \times (n_1 \times n_2)^{-1}], \quad (2.6b)$$

The isotopies of the $SU(2)$ symmetry were outlined in Section 1.9.

It was popularly believed in the 20th century that the $SO(3)$ symmetry is broken for the ellipsoidic deformations of the sphere. However, it is easy to prove that $\hat{SO}(3)$ is isomorphic to $SO(3)$, e.g., because, in the transition from a Lie symmetry to its isotopic covering, the original generators, parameters and structure constants remain unchanged.

Conceptually, this is due to the fact that ellipsoid deformations of the semiaxes of the perfect sphere are compensated on isospaces over isofields by the *inverse* deformation of the related unit

$$Radius \quad 1_k \rightarrow 1/n_k^2, \quad Unit \quad 1_k \rightarrow n_k^2. \tag{2.7}$$

resulting in the reconstruction of the perfect sphere on isospace called the *isosphere*,

$$\hat{r}^{\hat{2}} = \hat{r}_1^{\hat{2}} + \hat{r}_2^{\hat{2}} + \hat{r}_3^{\hat{2}}. \tag{2.8}$$

with consequential reconstruction of the exact rotational symmetry.

Alternatively, we can say that the reconstruction of the exact rotational symmetry is due to the structure of the basic invariant given by

$$L^2 = (length)^2 \times (unit)^2 \tag{2,9}$$

Consequently, a change of lengths joint with the inverse change of units leaves the invariant unchanged.

Similarly we have the reconstruction of the exact isospin symmetry in nuclear physics under electromagnetic interactions via the simple mechanism of embedding all symmetry breaking terms in the isounit [34].

(2) **Regular Lorentz-Santilli isotransforms** $\hat{SO}(3.1)$, first identified in Ref. [35] here presented for simplicity in the (3-4)-plane (see monograph [13b] for the general case)

$$x^{1'} = x^1, \quad x^{2'} = x^2, \tag{2.10a}$$

$$x^{3'} = \hat{\gamma} \times (x^3 - \hat{\beta} \times \frac{n_3}{n_4} \times x^4), \tag{2.10b}$$

$$x^{4'} = \hat{\gamma} \times (x^4 - \hat{\beta} \times \frac{n_4}{n_3} \times x^3), \tag{4.10c}$$

where

$$\hat{\beta}^2 = \frac{v_3^2/n_3^2}{c_o^2/n_4^2}, \quad \hat{\gamma} = \frac{1}{\sqrt{1-\hat{\beta}^2}}. \tag{2.11}$$

The isotopies of the spinorial covering of $SO(3.1)$ were studied or the first time in Ref. [38]. (see also monograph [13b]).

Again, it was popularly believed in the 20th century that the Lorentz symmetry is broken for deformations of the light cone. By contrast, Ref. [37] proved that the Lorentz symmetry does remain exact under the deformations of the light cone, provided it is treated with the appropriate mathematics. This result was achieved via the proof of the local isomorphism between $\hat{SO}(3.1)$ and $SO(3.1)$.

Conceptually, this is due to the reconstruction of the exact light cone on isospace over isofields called the *light isocone.* In fact, jointly with the deformation of the light cone

$$x^2 = x_3^2 - t^2 \times c^2 = 0 \rightarrow \frac{x_3^2}{n_3^2} - t^2 \times \frac{c^2}{n^4} = 0, \tag{2.12}$$

we have the corresponding inverse deformations of the units, thus reconstructing the original light cone on isospaces over isofields,

$$\hat{x}^{\hat{2}} = \hat{x}_3^{\hat{2}} - \hat{t}^{\hat{2}} \times c^{\hat{2}} = 0. \tag{2.13}$$

The reader should be aware that the above reconstruction includes the preservation on isospace over isofields of the original characteristic angle of the conventional light cone. Consequently, *the maximal causal speed on isospace over isofields is the conventional speed of light in vacuum c.*

The understanding of the isoscattering theory requires the knowledge that *in the transition from the exterior to the interior region (as depicted in Figure 2 of Paper I), the speed of light and related light cone remain unchanged, and only their realizations change.*

(3) **Regular isotranslations** $\hat{\mathcal{T}}(4)$, first studied in ref. [37] (see monograph [13b] for the general case). can be expressed with the following lifting of the conventional translations $x^{\mu'} = x^\mu + a^\mu, \mu = 1,2,3,4,$ and a^μ constants,

$$x^{\mu'} = x^\mu + A^\mu(a, \ldots), \tag{2.14}$$

where

$$A^\mu = a^\mu (n_\mu^{-2} + a^\alpha \times [n_\mu^{-2} \hat{,} P_\alpha]/1! + \ldots), \tag{2.15}$$

and there is no summation on the μ indices.

Readers should note the highly nonlinear character of the above translations. Nevertheless, the components of the linear momentum isocommute,

$$[P\mu \hat{,} P\nu] = P\mu \times T \times P\nu - P\nu \times T \times P\mu = 0, \ \mu, \nu = 1, 2, 3, 4. \tag{2.16}$$

while the generators of nonlinear transformations (2.14)-(2,15) do not commute in conventional space,

$$[P\mu, P\nu] = P\mu \times P\nu - P\nu \times P\mu \neq 0, \ \mu, \nu = 1, 2, 3, 4. \tag{2.17}$$

As we shall indicate in the next section, the above occurrence illustrates the capability by isorelativity of turning *curved* spaces into an equivalent *isoflat space* [40] with nontrivial implications, such as a consistent operator formulation of gravity, new grand unification, new interior gravitational models and other advances.

Recall that the conventional scattering theory is based on the conventional Minkowski space and, therefore, has no gravitational contribution of any type. By contrast, the above features imply that *the isoscattering region is isoflat and, therefore, with a primary gravitational, Finslerian and other contributions.*

4) **Regular isodilations** $\hat{\mathcal{T}}(1)$, first identified in Ref. [37] (see, again, monograph [13b] for the general case)

$$\hat{\eta} \to \hat{\eta}' = w^{-1} \times \hat{\eta}, \ \hat{I} \to \hat{I}' = w \times \hat{I}, \tag{2.18a}$$

$$\begin{aligned}(x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} &\equiv [x^\mu \times (w^{-1} \hat{\eta}_{\mu\nu}) \times x^\nu) \times (w \times \hat{I}) = \\ &= (x^\mu \times \hat{\eta}'_{\mu\nu} \times x^\nu) \times \hat{I}', \ w \in \mathcal{R}.\end{aligned} \tag{2.18b}$$

As one can see, *the LPS isosymmetry is eleven dimensional,* the 11-th dimensionality being given by the new *invariance* under isodilations similar to that of the Hilbert space, Eq. (3.19) of Paper I. Contrary to popular beliefs in the 20th century, *the conventional Lorentz-Poincaré symmetry is* *also eleven dimensional,* trivially, because isoinvariance (2.18) also applies to the standard case.

Readers should keep in mind the transition from the conventional dilations, often requiring an enlargement of the spacetime dimensions, to their invariant formulation under isotopies within the conventional (3.1)-dimensions.

Predictably, the discovery of a new symmetry for the conventional spacetime has momentous implications. In fact, isosymmetry (2.18) was instrumental for the first known axiomatically consistent operator formulation of gravity [44], grand unification of electroweak and gravitational interactions [45,46], and other basic advances.

The reason that the eleventh dimensionality was not discovered until Ref. [37] of 1993 should not be surprising since the new isosymmetry required the prior discovery of *new numbers,* the isonumbers with arbitrary units [6].

The resulting eleven-dimensional regular LPS isosymmetry can be written

$$\hat{P}(3.1) = [\hat{SO}(3.1)\hat{\times}\hat{\mathcal{T}}(3.1)] \times \hat{\mathcal{C}}(1), \tag{2.19}$$

where $\hat{\times}$ is the direct isoproduct and $\times$ is the Kronecker product.

Readers should finally keep in mind the “direct universality” of the LPS symmetry indicated in Section 2.1 since said symmetry provides the universal *invariance* (rather than covariance) of all infinitely possible line elements (2.3) that include as particular cases the Riemannian, Finslerian or other line elements in $(3+1)$-dimensions [41-43].

the isotopies of the spinorial covering of the LP symmetry will be indicated in Section 3 jointly with the isotopies of Dirac’s equation.

2.4. Deformations-Isotopies of Special Relativity. As indicated in Section 2.1, a central objective of the studies here considered is the preservation of Einstein axioms for interior conditions and the enlargement of their applicability via broader realizations. This objective was achieved first in paper [35] of 1983. Comprehensive elaborations were presented in monographs [12] of 1991 that include the isotopies of Galilei relativity we cannot possibly review here. Additional presentations are available in monographs [13] of 1995 and [16] of 2008.

As a result of these studies, *special relativity and its covering isorelativity can be presented at the abstract mathematical level via the same equations and axioms, merely submitted to different realizations, with a similar result holding between Galilei's relativity and its isotopic covering,* as illustrated, e.g., by the abstract identity of the light isocone with the conventional cone, Eq. (2.13), the preservation on isospacetime of the speed of light in vacuum as the maximal causal speed, and other features.

Regrettably, we cannot possibly review these studies. Nevertheless, it is evident that an abstract mathematical presentation would lead to possible misinterpretations in the applications of isorelativity to scattering processes or prevent them altogether. Therefore, for minimal self-sufficiency of these papers as well as to specialize the new laws to the interior of the scattering region, we recall the following isoaxioms in their *projection* on conventional spacetime restricted to the third space direction and time (see Refs. [13] for the general case):

ISOAXIOM I: The maximal causal speed within the scattering region is given by

$$\hat{V}_{max} = c \times \frac{n_3}{n_4} = C \times n_3. \tag{2.20}$$

ISOAXIOM II: The addition of speeds within the scattering region follows the isotopic law

$$V_{tot} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2} \times \frac{n_4^2}{n_3^2}}. \tag{2.21}$$

ISOAXIOM III: The dilation of time, the contraction of space and the variation of mass with speed within the scattering region follow the isotopic laws

$$t' = \hat{\gamma} \times t, \tag{2.22a}$$

$$\ell' = \hat{\gamma}^{-1} \times \ell, \tag{2.22b}$$

$$m' = \hat{\gamma} \times m, \tag{2.22c}$$

ISOAXIOM IV: The frequency shift within the scattering region follows the isotopic law

$$\omega' = \hat{\gamma} \times [1 - \hat{\beta}\ cos(\hat{\alpha}) \times \omega, \tag{2.23}$$

ISOAXIOM V: The mass-energy isoequivalence within the scattering region follows the isotopic law

$$\hat{E} = m\ V_{max}^2 = m \times c^2 \times \frac{n_3^2}{n_4^2} = m \times C^2 \times c_3^2. \tag{2.24}$$

A few comments are now in order. It should be stressed that, at the pure mathematical level, e.g., when formulated on isospace over isofields, the above isoaxioms coincide with conventional axioms. Consequently, *the isoscattering theory preserves Einstein's axioms everywhere in the exterior and the interior region, and merely uses a broader realization of the same axioms for the interior case.* As we shall see in the next section, we have the same occurrence in regard to the applicable mechanics because relativistic hadronic mechanics also coincides with relativistic quantum mechanics at the abstract mathematical level by conception and construction.

Interested readers are then suggested to inspect: the unique and unambiguous derivation of Isoaxioms I-V from the LPS isosymmetry [12,13,16], and the proof of their "direct universality" by J. V. Kadeisvili [41], A. K. Aringazin [42], and others. Therefore, *Isoaxioms I-V are the only known directly universal axioms for interior conditions that are invariant under a universal isosymmetry for arbitrary speeds of light.*

Rather unexpectedly, the above studies have established the *necessity of abandoning the speed of light as the maximal causal speed within physical media* because of unsurmontable difficulties in the use of the speed of light it in vacuum, the understanding being that the speed of light is indeed regained as maximal causal speed in empty space and isospace.

As an illustration, the use of special relativity within water leads to insufficiencies of one or the other axioms, e.g., if one assumes in water the maximal causal speed in vacuum, causality is preserved when electrons travel in water faster than the local speed of light, but the relativistic addition of speeds is violated. Vice-versa, the assumption of the speed of light in water as the maximal causal speed in water preserves the relativistic law of addition of speeds, but violates causality.

By comparison, isorelativity resolves these insufficiencies. In fact, water can be safely assumed as being homogeneous and isotropic. As a consequence, $n_3 = n_4$, and the maximal causal speed in water is the speed of light in vacuum, $V_{max} = c$, thus verifying causality, while the isorelativistic law of addition of speeds (4.21) is also verified [12,13,16].

A serious knowledge of the isoscattering theory requires a study of the experimental verification of isorelativity at the classical and operator levels presented in various papers and reviewed in monograph [16d], Chapter 5 of ref. [27] and paper 47. Some of the most important experimental verifications of relativistic quantum mechanics will be indicated in Section 3. An important, classical, verification has been that of Isoaxiom IV on the shift of light within physical media first presented in paper [47], and indicated in more details in the next section. Additional direct verifications are under way based on the repetition within physical media of the historical experimental verifications of special relativity all that were notoriously conducted in vacuum.

The specialization of isorelativity to the iso-Galilean case is an instructive exercise for the interested reader [12].

2.5. Implications for the Scattering Region. It is now important to identify the implications of isorelativity, specifically, for the interior of scattering processes because said implications will eventually be verified or dismissed by experiments in due time. Among a variety of rather intriguing implications, we identify the following ones:

2.5.1) **Isotime.** The time in the interior of the scattering region is no longer the same as our time, and it is given by a quantity known as *isotime* with vast physical and epistemological implications. In fact, the light cone is recovered identically on isospacetime, Eq. (2.13), but under the lifting of time into the isotime,

$$t \rightarrow \hat{t} = t/n_4. \tag{2.25}$$

By recalling that, according to all known fits of isorelativity, the index of refraction within hadronic matter is smaller than one, $n_4 < 1$ [16d,27], the reconstruction of the speed of light in vacuum as the maximal causal speed in interior conditions implies that *the scattering region is represented on isospacetime over isofields via an isotime generally in the future with* *respect to our time,*

$$\hat{t} = t/n_4 > t, \ n_4 < 1, \ C > c. \tag{2.26}$$

Readers with a technical knowledge of the Lie-Santilli isotheory will see that the replacement in the interior scattering region of our time with isotime is *necessary* to reach the universal invariance of the locally varying speed of light, since said invariance is based on the lifting of spacetime units, including that of time. Alternatively, the use of the isotime for interior conditions is necessary to avoid the inconsistency theorems.

2.5.2) **Isoshift,** For the case of null angle of aberration, and for $v = |v| << c$, the conventional Doppler's shift can be well approximated with the expression

$$\omega' = (1 \pm \frac{v}{c} + ...) \times \omega, \tag{2.27}$$

where the sign $-$ represents motion of the source *away* from the observer, in which case we have the *Doppler redshift,* and the sign $+$ represents motion of the source *toward* the observer, in which case we have the *Doppler blueshift.* It is evident that *special relativity predicts no shift of the frequency of light of any type for* $v = 0$,

The predictions of isorelativity are considerably different than the above. In fact, under the same assumptions as above, isolaw (2.13) can be written in good approximation

$$\omega' = (1 \pm \frac{v}{c} \times \frac{n_4}{n_3} + ...) \times \omega. \tag{2.28}$$

The most important implication, identified since 1991 [12], is that *isorelativity predicts a shift of the frequency of light even in the absence of any relative motion between the source, the medium and the observer,* called *isoshift.* This is due to the fact that the characteristic functions are generally dependent on the velocities (Section 2.2). Consequently, it is possible to have a dependence of the type $n_3/n_4 = v \times f(x, \xi, \omega, ...)$ under which expression (4.28) becomes

$$Lim_{v=0} \ \omega' = (1 \pm f(x, \xi, \omega, ...) + ...) \times \omega. \tag{2.29}$$

in which case *the frequency shift is no longer necessarily null for null relative speeds,* thus implying the following cases:

2.5.2A) **Santilli isoredshift.** It occurs for physical media of low density and has been experimentally verified in air, for certain astrophysical bodies, and other interior conditions [47]. This case essentially implies that *light loses energy $E = h\nu$ when propagating within physical media of low density, thus experiencing a decrease of its frequency without relative motion.* This possibility has been used in Ref. [47] to indicate the possible absence of universe expansion, big bang and dark matter since they all refer to motion of light within physical media in which the Doppler law not necessarily applies.

2.5.2B: **Santilli isoblueshift.** It is predicted for physical media of very high density, such as those in the interior of astrophysical bodies as well as in the interior of high energy scattering processes. In this case, we have the prediction that *light acquires energy $E = h\nu$ from the hyperdense medium, thus increasing its frequency.* Since all high energy scattering regions are hyperdense, *isorelativity predicts that photons detected outside the scattering region are isoblueshifted,* namely, they originate in the interior at a frequency smaller than that detected in the outside [47].

2.5.2C) **Doppler-Santilli isoshift.** It occurs when light propagates within a transparent physical medium but there exist a relative motion between the source, the medium and the observer, which relative motion remains fully relativistic. Note that there are several cases in which *the total shift can be null even though there exist a relative motion between the source and the observer,* since both, Doppler's and Santilli's shifts can be positive or negative depending on the case at hand.

2.5.3) **Isoequivalence.** Maximal causal speeds bigger than c are prohibited by special relativity because at the value $v = c$ the conventional axioms diverge, as it is the case for the mass

$$: im_{v=c} m' = Lim_{v=c} \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} = \infty. \tag{2.30}$$

This is no longer the case for isorelativity because it does not predict infinities at $v = c$ due to the indicated dependence of the characteristic quantities on the speed, for which we may have the expression for the mass

$$Lim_{v>c}\, m' = \frac{m}{\sqrt{1 - f(x, \xi, ...)}}. \tag{2.31}$$

Consequently, *Santilli isorelativity allows within physical media maximal causal speeds bigger than the speed of light in vacuum,* with far reaching implications.

For instance, Einstein's historical energy equivalence $E = m \times c^2$ was limpidly referred to *point-like particles in vacuum,* under which conditions it has received vast experimental verifications. However, the same equivalence principle has never received experimental verifications for extended and hyperdense particles, for which it is merely assumed as being valid, again, since special relativity does not allow any alternative formulation.

By contrast, isorelativity allows unlimited maximal causal speeds and the LPS isosymmetry uniquely and unambiguously predicts isoequivalence (2.24). By recalling again that all fits of experimental data available to date suggest causal speeds inside hadronic media bigger than c, *the energy equivalence of the scattering region predicted by isorelativity is bigger than that predicted by the conventional scattering theory,*

$$\hat{E} = m \times c^2 \times \frac{n_3^2}{n_4^2} > m \times c^2, \tag{2.32}$$

It should be indicated that isoequivalence (2.32) has been used in Ref. [47] to indicate the possible absence of dark energy due to the prediction of a total energy in the universe much bigger than that predicted by special relativity, since the maximal causal speed in the interior of black holes and other astrophysical bodies is predicted as being much bigger than c. The possible verification in scattering experiments of the isoblueshift would evidently confirm isoequivalence (2.24) and, therefore, the absence of dark energy.

2.5.4) **Isoparticles.** One of the most insidious misrepresentations for the isoscattering theory is the use of the conventional notion of particle. As it is well known, particles are generally referred to as irreducible unitary representations of the LP symmetry. However, such a symmetry is assumed as being inapplicable for the interior of the scattering region by central assumption, thus leading to major inconsistencies when using the conventional notion of particle under isotopies. Rather insidiously, these inconsistencies often remain undetected by non-experts in the new field.

Another implication of isospacetime mutation is that, *in the transition from motion in vacuum to motion within hyperdense scattering regions, particles experience a mutation of their "intrinsic characteristics," also called isorenormalization.resulting in the notion of isoparticles characterized by irreducible isounitary representations of the covering LPS isosymmetry* [12,13,16]. The covering notion of isoparticle requires an in depth knowledge of *regular and irregular isorepresentations of Lie-Santilli isoalgebras,* that have been illustrated in Section 1.9 for the case of spin 1/2.

In essence, conventional renormalizations, those characterized by action-at-a-distance, potential interactions, leave unchanged the intrinsic characteristics of particles. However, when passing to the broader non-Hamiltonian interactions, their nonunitary structure causes an isorenormalization of all characteristics of particles, including intrinsic features, via the mechanism of alteration of Hamiltonian eigenvalues pointed out in Paper I, and more technically identified in Section 1.9 as occurring under irregular isosymmetries. Note that isorenormalizations are fully in line with isoshifts.

Experimentally, the evidence supporting the mutation of intrinsic characteristic of particles in interior conditions is rather vast, and includes particle physics. nuclear physics, chemistry and astrophysics [*loc. cit.*]. Theoretically, the quantitative study of mutations of intrinsic characteristics of particles is perhaps the most fascinating of the isoscattering theory because, contrary to popular beliefs in the 20th century, our knowledge of the irreducible representations of the abstract axioms of the Lorentz symmetry is at its infancy, as illustrated by their current vastly unknown irregular isorepresentations.

As we shall see in subsequent papers, the quantitative treatment of the isorenormalization of particles inside the scattering region is one of the most challenging problem for a serious appraisal of the isoscattering theory on mathematical, theoretical and experimental grounds.

2.5.5) **Isogravitation.** As indicated earlier, the Riemannian formulation of gravitation has no appreciable impact in the conventional scattering theory, and any attempt at its inclusion is faced with very serious consistency problems, such has the unavoidable lifting of the scattering theory to nonunitary forms, loss of quantum mechanics, lack of conservation over time of the numerical predictions, and other problems [48].

By contrast, a direct, axiomatically consistent impact of gravitation in scattering processes becomes unavoidable for the covering isoscattering theory. This important aspect can be initially seen from isoinvariant (2,4) showing that *the Minkowski-Santilli isogeometry and related isospacetimes includes as particular cases the (pseudo-) Riemannian, Finslerian and all other possible geometries and related spacetimes in* $(3+1)$*-dimensions* [37].

The transition from the conventional to the isotopic formulation of gravity is provided by the following steps first proposed in Ref. [44] of 1994

I) Factorizing any possible (nonsingular, pseudo-) Riemannian, Finslerian, or other metric $g(x,...)$ into the *Minkowskian* metric η and a 4×4 matrix $\hat{T}_{gr}(x,...)$,

$$g(x,...) = \hat{T}_{gr}(x,...) \times \eta, \tag{2.33}$$

II) Introducing the *gravitational isounit* as the *inverse* of the matrix $\hat{T}_{gr}(x,...)$,

$$\hat{I}_{gr}(x,...) = \hat{T}_{gr}(x,...)^{-1}. \tag{2.34}$$

III) Reconstructing the Minkowskian geometry, the LP symmetry and special relativity with respect to the above gravitational isounit.

Since $\hat{T}_{gr}(x,...)$ is necessarily positive-definite for all nonsingular Riemannian, Finslerian or other metrics in $(3+1)$-dimensions, the resulting LPS isosymmetry is isomorphic to the conventional LP symmetry, thus allowing the treatment of gravitation with all the formulations studied so far in these papers, as well as those we shall study in the future. The resulting formulation of gravity is today known as *Santilli isogravitation.*

As an illustration, the celebrated Schwarzschild line element in the coordinates (θ, ϕ, r, t) admits the following *identical* reformulation as the isometric in isospacetime

$$ds^2 = r^2(d\theta^2 + sin^2 d\theta^2 + d\phi^2) + (1 - \frac{2\times M}{r})^{-1} \times dr^2 -$$

$$-(1-\frac{2\times M}{r}\times dt^2\equiv\hat{T}_{sch}\times\eta\equiv\hat{\eta}, \tag{2.35}$$

with *gravitational isounit and isotopic element*

$$\hat{T}_{sch}=Diag.[1,1,(1-\frac{2\times M}{r})^{-1},(1-\frac{2\times M}{r})], \tag{2.36a}$$

$$\hat{I}_{sch}=Diag.[1,1,(1-\frac{2\times M}{r}),(1-\frac{2\times M}{r})^{-1}], \tag{2.36b}$$

where one should note the positive-definiteness of the gravitational isounit and we assume the reader is aware from Paper I of the need for isotrigonometry in the isotopic reformulation, hereon tacitly assumed.

The implications at large of the above formulation of gravitation are far reaching, and their specializations to scattering processes should be at least summarily outlined here due to their significance, such as the clear prediction presented in Paper III that very high energy scattering experiments can indeed generate mini-black-holes.

Let us begin our short outline with the following important

LEMMA 2.5.1 [40,44]: The isotopic reformulation of the Riemannian gravitation implies the loss of curvature in favor of the isoflatness of the Minkowski-Santilli isogeometry.

This fundamental result can be seen in a variety of ways, e.g., from the fact that, by conception and construction outlined in Section 2.2, the Minkowski-Santilli isogeometry is locally isomorphic to the *Minkowski* geometry, thus prohibiting any conventional notion of curvature. Alternatively, one can see the loss of curvature on a conceptual basis by noting that gravitation is entirely contained in the isotopic element $\hat{T}_{gr}$. Consequently, the deformation of the Minkowski metric caused by gravitation

$$\eta\ \rightarrow\ \hat{T}_{gr}\times\eta=\hat{\eta}, \tag{2.37}$$

is compensated by the *inverse* deformation of the unit

$$I=Diag.(1,1,1,1)\ \rightarrow\ \hat{I}_{gr}=(\hat{T}_{gr})^{-1}, \tag{2.38}$$

without altering the original flatness in view of the novel isodilation symmetry of the Hilbert space, Eq. (3.19) of Paper I, the new isodilation invariance (2.18), or the very structure (2.9) spacetime invariants. In turn the loss of curvature in favor of isoflatness has the following implications rather important for scattering processes:

2.5.5A) **Consistent operator form of gravitation.** As it is well known, a consistent operator formulation of the Riemannian gravitation acceptable by the scientific community at large has not been achieved in one century of efforts due to unsurmontable problematic aspects or sheer inconsistencies caused by curvature, the ensuing nonunitary character of the theory, lack of the PCT theorem and other problems [48]. By comparison, isogravitation admits an axiomatically consistent operator formulation first achieved in ref. [44] merely given by embedding gravity in the *unit* of relativistic quantum mechanics, thus preserving its abstract axioms, and ensuing consistency, including the correct formulation of the PCT theorem and all that. Note that this result *cannot* be achieved with the Riemannian curvature (see Ref. [16c] for details).

2.5.5B) **Universal invariance of gravitation.** As it is well known, the conventional Riemannian formulation of gravitation solely admits a '"covariance." But its structure is notoriously noncanonical, thus activating the theorems of catastrophic inconsistency (see Ref. [48] for details). By comparison, the isotopic formulation of gravity admits the universal LPS isoinvariance with the resolution of said inconsistency problems. Again, the reader should keep in mind that the invariance of gravitation is impossible with the Riemannian curvature (see Refs. [37.44.16c] for details).

2.5.5C) **Unification of the Minkowskian and Riemannian geometries.** Traditionally, the Minkowskian and Riemannian geometries are differentiated, as it should be, when formulated on conventional spaces over conventional fields. However, the use of isospaces over isofields has allowed the unification of these two geometries into one single geometry, the Minkowski-Santilli isogeometry, and their differentiation via different isounits first achieved in Ref. [40]. But the isometric $\hat{\eta}(x, ...)$ has an explicit dependence on coordinates and other variables. Consequently, the

Minkowski-Santilli isogeometry admits the entire machinery of the Riemannian geometry., such as covariant derivative., Christoffel's symbols, etc. only isotopically reformulated, with consequential *geometric unification of special and general relativities.* This result has the consequence, rather important for scattering processes, according to which *the Einstein-Hilbert field equations are preserved and identically reformulated in an invariant operator version for the interior of the scattering region.* Note again that this result would be inconsistent under a Riemannian curvature on a number of grounds [48].

2.5.5D) **Isotopic grand unification.** It is equally well known that a grand unification of electroweak and gravitational interactions in a form acceptable by the scientific community at large has escaped all efforts beginning with Einstein. It is today know that the difficulties originate from: A) Inconsistencies in unifying a theory possessing an invariance with another theory solely possessing covariance (due to the activation by the latter of the inconsistency theorems for the entire unification [44]); B) Inconsistencies in unifying an operator theory on a flat spacetime with another on a curved spacetime (due to the ensuing nonunitary structure and activation, again, of the inconsistency theorems); and C) Inconsistencies in unifying a theory with full democracy between particles and antiparticles with a gravitational theory insufficient for the description of antiparticles. e/g/., without any distinction whatsoever between neutral particles and antiparticles (see Section 2.5). Thanks to the removal of curvature and the achievement of an invariant operator formulation, isogravity has resolved insufficiencies A, B, C, resulting in an axiomatically consistent *iso-grand-unification* in which gravitation is embedded in the unit of electroweak theories, first achieved in Refs. [45,46] (see monograph [15] for a comprehensive presentation including the necessary gravitational treatment of neutral or charged antimatter).

2.5.5E) **Interior gravitation.** As indicated in Paper I, prior to Einstein's time, there was a clear differentiation between exterior and interior problems. In fact, Schwarzschild wrote *two* papers, the first one for the *exterior gravitational problem* with his historical metric (2.35) and a vastly ignored second paper on the *interior gravitational problem.* The distinction between exterior and interior problems was then ignored for about one century via the abstraction of the latter problems to isolated point-particles in vacuum. The No Reduction Theorems reviewed in Paper I have suggested a return to the full differentiation between exterior and interior gravitational problems, thus relegating metric (2.35) to the meaning intended by its originator, namely, for the exterior problem only. The advent of isogravitation has permitted significant advances in interior gravitational; problems, e.g., by achieving for the first time a direct geometric representation (that is, a representation via the isometric) of the locally varying speed of light, the density of the interior medium and other features. Gravitational collapse is then represented with the *limit of null value of the space component of the isounit and the limit to a divergent value of its time component,* as geometrically expected in any case, i.e.,

$$\hat{I}^{int}_{gr,space}(x, v, \xi, \omega, \psi, \partial\psi, ...) \rightarrow \infty, \tag{2.39a}$$

$$\hat{I}^{int}_{gr,time}(x, v, \xi, \omega, \psi, \partial\psi, ...) \rightarrow 0. \tag{2.39b}$$

To understand the above reformulation olf gravitational collapse, one should keep in mind that *isotopic rules (2.39) are equations that can be solved not only in the coordinates, as it is the case for the Schwarzschild metric (2.35), but also in the velocities and other variables as it is necessary for realistic models of interior gravitation.* The issue as to whether a true singularity such as the notion of *black hole,* is preserved by interior isogravitation, or we merely have a gravitational collapse without singularity, such as the notion of *brown hole,* is under study at this writing and the outcome will be reported in a future paper.

3. Deformations-Isotopies of Mechanics

3.1. Introduction. The central dynamical problem of the isoscattering theory is that at the foundation of hadronic mechanics, namely, the achievement of a consistent operator formulation of nonconservative systems, called in these papers non-Hamiltonian systems. We are here referring to a problem that remained unsolved for most of the 20th century physics, whose solution required several decades of trials and errors. We believe it is important to outline the main points of this scientific journey so as to avoid potential misrepresentation of the proposed isoscattering theory in the event based on the older literature, or the use of mechanics that have been proved as being inappliucable.

A considerable number of applied mathematicians, theoretical physicists and experimenmtalists have contributed to the deformations-isotopies of classical and quantum mechanics to represent non-Hamiltonian systems, including in chronological order from 1978 to 1983: R. M. Santilli, H. C. Myung, S. Okubo, J. Fronteau, A Tellez-Arenas, R. Mignani, A. O. E. Animalu, J. A. Kobussen, Y. Ylamed, N. Salingaros, T. Giill, A. J. Kalnay, H. Rauch, G. Eder, P. Caldirola, R. Trostel, A. Schober, R. J. Slobodrian, J. Sun, A. de Wet, A. D. Jannussis, G. Brodimas, D. S. Sourlas, N. Salingaros, N. Tsagas, D. P. K. Ghikas, E. Kapushik, F. Rohrlich, J. Snyaticki, N. Salingaros, P. Truini, G. Cassinelli, G. Lochak, D. Y. Kim, J. Salmon, M. Grmela, E. Tontio, J. G. Gilson, V. K. Agrawala, W. H. Steeb, M. Mijatovich, R. Broucke, and numerous others in the subsequent years. Regrettably, we cannot possibly review the studies by so many authors and have to restrict our outline to the aspects directly relevant for the isoscattering theory (see the comprehensive bibliography in Volume [16a]).

The insufficiency that prevented the achievement of a consistent operator formulation of non-Hamiltonian systems for such a long time was the absence of their universal representation via an action principle. due to the crucial role of the latter for a consistent map to operator forms.

The scientific journey initiated with Volume [10a] in which it was established that non-Hamiltonian systems in two or more dimensions generally violate the integrability conditions for the existence of a Hamiltonian, the *conditions of variational selfadjointness.* Consequently, non-Hamiltonian systems in more than one dimension do not admit an analytic representation via the conventional *Hamiltonian action principle.*

The scientific journey then continued with Volume [10b] achieving the '"direct universality" for the representation of all well behaved non-Hamiltonian systems via the *Birkhoffian action principle.* However, the latter action principle resulted as being inapplicable for the needed operator map for various reasons, such as the fact that the emerging "wavefunctions" would depend on both coordinates and momenta, $\psi = \psi(t, r, p)$, thus having a structure beyond our current knowledge of operator mechanics. and departing in any case from the abstract axioms of quantum mechanics.

Following the abandonment of the Birkhoffian mechanics, numerous additional mechanics were attempted (as listed, e.g., in the 1991 edition of Refs. [13]), but they all had a noncanonical time evolution formulated on conventional spaces over conventional fields, thus activating the inconsistency theorems of Paper I.

The needed breakthrough finally occurred with the discovery in memoir [7] of 1996 of the *isodifferential calculus* that permitted the achievement of the needed axiom preserving isotopies of the classical Hamiltonian mechanics with a consistent map to the operator image that resulted indeed to be an isotopy of quantum mechanics, as desired.

There was a similar impass in the construction of hadronic mechanics. Following the original proposal [5] with the basic Lie-isotopic and Lie-admissible dynamical equations, all possible isotopies and genotopies of the Hilbert space, functional analysis, Lie algebras, etc., were reached by the early 1980s. Nevertheless, hadronic mechanics failed to be invariant, thus verifying the inconsistency theorems of nonunitary theories. Additionally, and perhaps most insidiously, hadronic mechanics missed consistent isotopies and genotopies of the Schrödinger realization of the linear momentum, thus prohibiting practical applications, e.g., because of the lack olf a consistent formulation of the angular momentum.

The needed breakthrough occurred, again, with memoir [7] thanks to the discovery of the isotopies and genotopies of the differential calculus that finally permitted, after two decades of failed attempts, the achievement of consistent isotopies and genotopies of the linear and angular momenta, with consequential basic invariance over time.

In conclusion, readers interested in a serious study of the isoscattering theory should be aware that the various classical and operator mechanics existing in the literature prior to the generalized differential calculi of Ref. [7] are either inapplicable to scattering processes, or they verify the theorems of catastrophic inconsistencies.

3.2. Deformations-Isotopies of Newtonian Mechanics. The isoscattering theory is based on the *isotopic branch of hadronic mechanics,* also known as *isomechanics* [4,5, 13,16]. Its primitive notion is that of *closed-isolated non-Hamiltonian systems* introduced in monograph [10b] defined as systems with Hamiltonian (*variationally selfadjoint,* SA) and non-Hamiltonian (*variationally nonselfadjoint,* NSA) internal forces, yet verifying the conventional ten total conservation laws of the Galilei and Lorentz-Poincaré symmetries. We are referring to the following class of Newtonian systems

$$m_k \times \frac{dv_k}{dt} - F_k^{SA}(t, r) - F_k^{NSA}(t, r, v, ...) = 0, \quad k = 2, 3, 4, ..., N, \qquad (3.1)$$

whose NSA forces verify the closure conditions

$$\sum_k \mathbf{F}_k^{NSA} = 0, \qquad (3.2a)$$

$$\sum_k \mathbf{r}_k \bigodot \mathbf{F}_k^{NSA} = 0, \qquad (3,2b)$$

$$\sum_k \mathbf{r}_k \bigwedge \mathbf{F}_k^{NSA} = 0, \qquad (3.2c)$$

with trivial relativistic extension, under which the conventional ten conservation laws are automatically verified.

Prior to Refs. [10], it was popularly believed that only Hamiltonian system=s verify the conventional ten conservation laws. Refs. [10] dispelled this unsubstantiated belief by showing that interior dynamical problems of isolated systems, such as Jupiter, do indeed include contact, nonconservative, non-Hamiltonian internal forces, as established by nature in any case.

Hence. it is important for this paper to clarify that the basic assumption of the isoscattering theory, the admission of contact non-Hamiltonian forces for the interior scattering region, originates at the primitive Newtonian level, as necessary from the *No Reduction Theorems* reviewed in Paper I.

Note that the verification of conventional total conservation laws is crucial for isomechanics and the isoscattering theory. In fact, when the system is open we have the broader *Lie-admissible genomathematics and genomechanics* [13,16].

The next step is the representation of systems (3.1) with the isotopic lifting of Newton's equations, first introduced in Refs. [7,13b] and known as *Newton-Santilli isoequations,* permitting the first known broadening of

Newton's equation since Newton's time for the representation of extended, nonspherical and deformable particles under SA and NSA forces.

Far from being trivial, the referred structural broadening of Newton's equation required a structural generalization of the entire mathematics of Newton's equations, including numbers, vector spaces, functional analysis, as well as the very differential calculus discovered by Newton himself (with Leibnitz).

Despite its diversification, this vast effort remained insufficient for the desired structural broadening of Newton's equation, that for the representation of *extended* particles, since such a representation is manifestly absent in Newton' s formulation. In fact, the background Euclidean geometry solely permits the representation of particles as being point-like. Maturity was finally achieved thanks to the construction by the mathematicians Gr. Tsagas, D. Sourlas, R. Falcon Ganfornina and J. Nunez Valdes of the all fundamental *isotopology* [29] that finally allowed the consistent representation of extended particles beginning at the primitive topological level.

The *Newton-Santilli isoequations*, first reached in Ref. [7], can be expressed for simplicity in the following form with the sole isodifferential in the velocities (see Ref. [13b] for the general form)

$$m \times \frac{\hat{d}\hat{v}}{\hat{d}t} - F^{SA} = m \times \frac{dv}{dt} - F^{SA} - F^{NSA} = 0. \tag{3.3}$$

with elementary solution for v constant and F^{NSA} not dependent explicitly on time (see [*loc. cit.*] for the general case)

$$\hat{I}_t = 1, \quad \hat{t} = t, \quad \hat{I}_r = 1, \quad \hat{r} = r, \tag{3.4a}$$

$$\hat{I}_v = exp(\frac{t}{mv} F^{NSA}), \quad \hat{v} = v \times \hat{I}_v. \tag{3.4b}$$

It is important for this paper to know that the main mechanism of the isoscattering theory, the embedding of the non-Hamiltonian interactions in a generalization of the basic unit, originates at the primitive Newtonian level. Note the emergence of a realization of the isounit via the exponent, an occurrence that will result as being rather general for the isoscattering theory, as shown in Paper III.

Note finally the mechanism of the representation of non-Hamiltonian Newtonian systems consisting in turning their NSA form when formulated on conventional Euclidean space over conventional fields, into an *identical,* fully SA form when formulated on Euclid-Santilli isospaces over isofields. Such an identical SA form is then at the foundation of the subsequent representation via an isoaction principle and ensuing consistent map to operator isomechanics. Intriguingly, the achievement of this new representation required a generalization of the very differential calculus Newton had to develop for his original formulation.

3.3. Deformations-Isotopies of Hamiltonian Mechanics. the next central methodological branch of isomechanics is the analytic representation of Eqs. (3.4) via a *variational isoaction principle* also introduced in Refs. [7,13], here expressed in its general form for space and time isotopies

$$\hat{\delta}\hat{A}^o = \hat{\delta}\int_{t_1}^{t_2}(\hat{p}_k \hat{\times} \hat{d}\hat{r}^k - \hat{H} \hat{\times} \hat{d}\hat{t}) = \hat{\delta}\int_{t_1}^{t_2}[p_k \times \hat{T}^k_{\hat{r},i}(t,r,p,...) \times \hat{d}\hat{r}^i - \hat{H} \times \hat{T}_{\hat{t}} \times d\hat{t}] = 0, \tag{3.5}$$

characterizing thhe *Hamilton-Santilli isoequations*

$$\frac{\hat{d}\hat{r}^k}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k}, \quad \frac{\hat{\partial}\hat{p}_k}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{r}^k}. \tag{3.6}$$

that provide a direct analytic representation of NSA Newtonian systems (3.4), as well as the *Hamilton-Jacobi-Santilli isoequations* [*loc. cit.*]

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{t}^d} + \hat{H} = 0, \tag{3.7a}$$

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{r}^k} - \hat{p}_k = 0, \tag{3.7b}$$

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{p}_k} \equiv 0. \tag{3.7c}$$

. The latter expression, evidently at the foundation of operator maps, illustrate again the fundamental role of the isodifferential calculus for all deformations-isotopies.

Note the formal identity at the abstract level of the conventional Hamiltonian mechanics and its isotopic image. This illustrates that the abstract axioms of Hamiltonian mechanics have representational capability dramatically broader than those believed for centuries, although they can be seen only under the using the appropriate broader mathematics.

3.4. Deformations-Isotopies of Quantum Mechanics. The conventional naive quantization

$$A^o = \int_{t_1}^{t_2} (p_k \times dx^k - H \times dt) \rightarrow -i \times \hbar \times log\psi, \tag{3.8}$$

is lifted into the following *Animalu-Santilli isoquantization* [49] via the use of the isologarithm, Eq. (3.5) of Paper I,

$$\hat{A}^o = \int_{t_1}^{t_2} (\hat{p}_k \hat{\times} \hat{d}\hat{x}^k - \hat{H} \hat{\times} \hat{d}\hat{t}) \rightarrow -i \times \hat{log}\hat{\psi} = -i \times \hat{I} \times Log\hat{\psi}, \tag{3.9}$$

with corresponding isotopies for the symplectic and other operator maps [13].

We are now equipped to present the operator image of the classical isotopies. The first nonunitary image of the Schrödinger equation was presented in the original proposal [5] to build hadronic mechanics, and then studied by several authors, such as Myung and Santilli [50], Mignani [51] and others (see the general bibliography in Ref. [16a]), although all these initial versions were formulated on conventional Hilbert spaces and/or over conventional fields, thus activating the inconsistency theorems reviewed earlier.

The axiomatically correct, and invariant, nonunitary isotopic image of Schrödinger equation was reached in Ref. [7] by applying map (3.9) to Eqs./ (3.7). The resulting equations are today known as *Schrödinger-Santilli isoequations* on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$ and can be written (see Refs. [13] for a detailed treatment)

$$\hat{i} \hat{\times} \hat{\partial}_{\hat{t}} \hat{\psi}(\hat{t}, \hat{r}) = \hat{H} \hat{\times} \hat{\psi}(\hat{t}, \hat{r}) = \hat{E} \hat{\times} \hat{\psi}(\hat{t}, \hat{r}), \tag{3.10}$$

with *isolinear momentum,* first formulated in 1995 [*loc. cit.*]],

$$\hat{p}_k \hat{\times} \hat{\psi}(\hat{t}, \hat{r}) = -\hat{i} \hat{\times} \hat{\partial}_k \hat{\psi}(\hat{t}, \hat{r}), \tag{3.11}$$

canonical isocommutation rules

$$[\hat{r}^i \hat{,} \hat{p}_j] = \hat{i} \hat{\times} \hat{\delta}^i_j, \quad [\hat{r}^i \hat{,} \hat{r}^j] = [\hat{p}_i \hat{,} \hat{p}_j] = \hat{0}. \tag{3.12}$$

isonormalization

$$< \hat{\psi} | \hat{\times} | \hat{\psi} > \times \hat{I} = \hat{I}, \tag{3.13}$$

isoexpectation values of an iso-Hermitean operator $\hat{A}$

$$\hat{<} \hat{A} \hat{>} = < \hat{\psi} | \hat{\times} \hat{A} \hat{\times} | \hat{\psi} > \times \hat{I}, \tag{3.14}$$

and *isounit identities*

$$\hat{I} \hat{\times} \hat{\psi} = \hat{\psi}, \quad < \hat{\psi} | \hat{\times} \hat{I} \hat{\times} | \hat{\psi} > = \hat{I}. \tag{3.15}$$

assuring that $\hat{I}$ is indeed the correct basic unit of the theory (see Section 3 of Paper I for details).

The above equations are written on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$ and can be written in their *projection* on $\mathcal{H}$ over C

$$i \times \hat{I}_t(t, r, p, \xi, \omega, \psi, \partial\psi, ...) \times \partial_t \hat{\psi}(t, r) =$$

$$= H \times T_r(t, r, p, \xi, \omega, \psi, \partial\psi, ...) \times \hat{\psi}(t, r) = E \times \hat{\psi}(t, r), \tag{3.16a}$$

$$p_k \times T_r(t, r, p, \xi, \omega, \psi, \partial\psi, ...) \times \hat{\psi}(t, r) =$$

$$= -i \times \hat{I}^i_k((t, r, p, \xi, \omega, \psi, \partial\psi, ...) \times \partial_i \hat{\psi}(t, r), \tag{3.16b}$$

$$[r^i \hat{,} p_j] = i \times \hat{I}_r \times \delta^i_j, \quad [r^i \hat{,} r^j] = [p_i \hat{,} p_j] = 0. \tag{3.16c}$$

$$\hat{<} \hat{A} \hat{>} = < \hat{\psi} | \times T \times A \times T \times | \hat{\psi} > \times \hat{I}. \tag{3.16d}$$

We also have the following *isoplanewaves,* namely, conventional planewaves experiencing a mutation due to their immersion within the scattering region,

$$\hat{\psi}(r) = \hat{e}^{i \times k \times r} = \hat{I} \times (e^{i \times k \times T \times r}), \tag{3.17}$$

with isoeigenvalue equation (assuming for simplicity that the isounit does not depend on coordinates, see Ref. [13b] for the general case)

$$p \times T \times \hat{\psi} = -i \times \hat{I} \times \partial_r \hat{\psi} = -i^2 \times \hat{I} \times T \times k \times \hat{\psi} = k \times \hat{\psi}, \qquad (3.18)$$

The reader can now see the fundamental relevance for hadronic mechanics of the isodifferential calculus because, until achieved, hadronic mechanics had no consistent formulation of the linear momentum, angular momentum, plane waves and other basic features.

The isotopies of Heisenberg equations were introduced in the original proposal [5] of 1978; studied by various authors, such as Myung and Santilli [51] and others (see Refs. [16a] for comprehensive bibliography); then reformulated in Ref. [7] via the isodifferential calculus; and are today known as *Heisenberg-Santilli isoequations.* They can be written in their general form on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$ for the finite time evolution of a (Hermitean) operator $\hat{A}$

$$\hat{A}(\hat{t}) = \hat{U}(\hat{t}) \hat{\times} \hat{A}(\hat{0}) \hat{\times} \hat{U}^{\dagger}(\hat{t}) = (\hat{e}^{i \times H \times t}) \hat{\times} \hat{A}(\hat{0}) \hat{\times} (\hat{e}^{-i \times t \times H}), \qquad (3.19)$$

characterizing a one-dimensional *Lie-Santilli isotransformation group* (see Section 1.5) with corresponding infinitesimal form

$$\hat{i} \hat{\times} \frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} = [\hat{A} \hat{,} \hat{H}]. \qquad (3.20)$$

By using isoexponentiation (3.4), the above equations can be written in their projection on $\mathcal{H}$ over $\mathcal{C}$

$$A(t) = e^{i \times H \times T \times t}) \times A(0) \times (e^{-i \times t \times T \times H}), \qquad (3.21a)$$

$$i \times \hat{I}_t((t, r, p, \xi, \omega, \psi, \partial\psi, ...) \times \frac{dA}{dt} =$$

$$= A \times T_r((t, r, p, \xi, \omega, \psi, \partial\psi, ...) \times H - H \times T - r((t, r, p, \xi, \omega, \psi, \partial\psi, ...) \times A =$$

$$= [A \hat{,} H], \qquad (3.21b)$$

with fundamental isounitary property

$$\hat{U} = \hat{e}^{i \times H \times t}, \quad \hat{U} \hat{\times} \hat{U}^{\dagger} = \hat{U}^{\dagger} \hat{\times} \hat{U} = \hat{I}_r. \qquad (3.22)$$

This concludes our elementary review of the basic equations of the nonrelativistic isotopic branch of hadronic mechanics as minimally needed for the elaboration of the isoscattering theory. The elements of the relativistic extension are indicated in the next section.

We should recall from Section 3.5 of Paper I that isotopies preserve Hermiticity to the extent that the operations of Hermiticity and iso-Hermiticity coincide. Hence, all observables of quantum mechanics remain observables for the covering hadronic mechanics. Also, by conception and construction, quantum and hadronic mechanics coincide at the abstract, realization-free level. Therefore, any criticism on the axiomatic structure of hadronic mechanics is indeed a criticism on the axiomatic structure of quantum mechanics.

Additionally, we should also recall that *hadronic mechanics has been conceived and constructed as a kind of completion of quantum mechanics much along the historical Einstein-Podolsky-Rosen argument* (see Ref. [34] on the latter aspects. In this way, *the isoscattering theory can be considered a realization of one of the possible realizations of hidden variables with explicit and concrete identification of their realization, physical origin and meaning.*

By recalling the content of Paper I, isounitarity property (3.22), as well as the isolocality and isolinearity, establish the resolution of the inconsistency theorems for nonunitary theories. In particular, the isolinearity is crucial for the application of the mechanics to multiple scattering process, an occurrence with unresolved problematic aspects for a nonlinear formulation of the conventional scattering theory (see also Sections 3.4 and 3.5 of Paper I).

Note that the various "deformations of quantum mechanics" existing in the literature were formulated years following the appearance of the isotopies and genotopies [4,5] of 1978, and are essentially *identical* to these original formulations formulated on conventional spaces over conventional fields, thus being catastrophically inconsistent for the reasons indicated earlier (see Refs. [6-12] of Paper I).

3.5. Deformations-Isotopies of Dirac's Equations. As it is well known, Dirac's equation plays a central role in Feynman's diagrams. Consequently, the the covering isoscattering theory is crucially dependent on the deformations-isotopies of Dirac's equation.

Due to their importance, these isotopies were studied since the early stages of hadronic mechanics (see Ref. [12,13b] for original studies). In these papers, we shall use the first invariant isotopies of Dirac's equation achieved in Refs. [38,39] and used for the construction of the isotopies of the spinorial covering of the LP symmetry, as well as for the first known relativistic representation of all characteristics of the neutron in its synthesis from a proton and an electron in the core of a star (see Kaldeivili review [52]). The new equations are today known as the *Dirac-Santilli isoequations,* where we use the plural due to the variety of different realizations characterized by different isounits.

As it is also well known, the conventional Dirac equation represents *an electron in the external field of the proton.* The primary use of the covering isotopic formulation is that of representing a *mutated (isorenormalized) electron when immersed inside the proton also assumed as external.* The mutated electron was called *eleton* in the original proposal [5], but the name of *isoelectron* is nowadays more generally adopted due to the use of the prefix iso for protons and other particles when in interior conditions.

The isotopies of the Dirac equation belong to the branch of hadronic mechanics known as *relativistic isomechanics* (see memoir [53] for a review). Its foundations are given by the second-order iso-Casimir invariant (2.5b) that, via the use of the relativistic version of the isolinear momentum (3.11) characterizes the *isotopies of the Klein-Gordon equation* on $\hat{M}(\hat{x}, \hat{\eta}, \hat{\mathcal{R}})$ (see Ref. [13b] for detailed treatment of relativistic hadronic mechanics)

$$(\hat{\Xi}^{\mu\nu}\hat{\times}\hat{p}_\mu\hat{\times}\hat{p}_\nu - \hat{m}^{\hat{2}}\hat{\times}\hat{V}_{max}^{\hat{2}})\hat{\times}|\hat{\psi}(x)>=$$
$$= (\hat{\eta}^{\mu\nu} \times \hat{\partial}_\mu\hat{\partial}_\nu - m^2 \times V_{max}^2) \times |\hat{\psi}(x)>= 0. \tag{3.23}$$

as well as *relativistic isokinematics.*

The desired isotopies of Dirac's equation are then obtained via a linearization of the above equation similar to the conventional linearization, resulting in the following two cases:

CASE I: REGULAR DIRAC-SANTILLI ISOEQUATIONS.
In this case, the regular isotopies of spacetime can be described by the isounit, isotopic element and isometric (2.1) hereon denoted $\hat{I}_{st}$, T_{st}, and

$\hat{\Xi}_{st} = \hat{\eta}_{st} \times \hat{I}_{st}$, respectively, while the regular isotopies of the spin are tose of Section 1.9 with isounit, isotopic element and isometric (1.33) hereon denoted $\hat{I}_{sp}$, T_{sp} and $\hat{\delta}_{sp}$.respectively.

By using again the notation of Eqs. (2.1b) and by assuming $\Gamma = \gamma \times \hat{I}_{st}$, the *regular Dirac-Santilli isoequations* can then be written [37.38]

$$(\hat{\Xi}^{\mu\nu}_{st} \hat{\times} \hat{\Gamma}_{\mu} \hat{\times} \hat{p}_{\nu} + \hat{i} \times \hat{m} \hat{\times} \hat{V}_{max}) \times \hat{T}_{st} \times |\hat{\psi}(x) >=$$

$$= (\hat{\eta}^{\mu\nu}_{st} \times \hat{\gamma}_{\mu} \times \hat{\partial}_{\nu} + i \times m \times V_{max}) \times |\hat{\psi}(x) >= 0. \tag{3.24}$$

where, by using isomatrices (1.36), the *regular isogamma matrices* have a structure of the type

$$\hat{\gamma}_k = b_k \times \begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \quad \hat{\gamma}_4 = i \times b_4 \times \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \tag{3.25}$$

with anti-isocommutation rules

$$\{\hat{\gamma}_\mu \hat{,} \hat{\gamma}_\nu\} = \hat{\gamma}_\mu \times T_{st} \times \hat{\gamma}_\nu + \hat{\gamma}_\nu \times T_{st} \times \hat{\gamma}_\mu = 2 \times \hat{\eta}_{\mu\nu}, \tag{3.26}$$

The first implications for the isoscattering theory is that of *embedding gravitation directly in the basic metric $\hat{\eta}$ of the scattering region.* As an example, rules (3.26) can characterize (twice) the Schwarzschild metric as in Eqs. (2.35). However, the isometric can be more complex than that to include velocity-dependent internal effects, as well as the locally varying speed of light.

As one can see, in this particular case, the isotopies essentially offer *five* additional characteristic quantities, four for the spacetime mutation and one for the spin, that are available for the representation of experimentally measurable features of the scattering region, such as shape, density and anisotropy (requiring precisely five values). Note that the representation of these features is essentially outside the capabilities of quantum scattering theories.

Reader can now see the comments on antimatter of Section 2.5 of Paper I, to the effect that *there exist no irreducible or reducible representation of the conventional $SU(2)$ spin algebra having the structure of Dirac's gamma matrices.* By contrast, the conventional gamma matrices are characterized by the Kronecker product of an irreducible representation for spin 1/2 and its isodual, since the isodual unit $I^d_{2\times 2} = -I_{2\times 2}$ appears directly in γ_4 and Pauli's matrices verify the isodual rule $\sigma^d_k = -\sigma^\dagger_k = -\sigma_k$.

Consequently, *the conventional Dirac equation is reinterpreted as representing one electron and its antiparticle* without any need of the "hole theory" or second quantization, since the isodual theory of antimatter holds at the *Newtonian* level, let alone that of first quantization (see monograph [15] for comprehensive studies).

Evidently, the above features persist under isotopies, and we shall write in these papers the isogamma matrices in the form

$$\hat{\gamma}_k = b_k \times \begin{pmatrix} 0 & \hat{\sigma}_k \\ \hat{\sigma}^d_k & 0 \end{pmatrix}, \quad \hat{\gamma}_4 = i \times b_4 \times \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & I^d_{2\times 2} \end{pmatrix}, \tag{3.27a}$$

$$|\hat{\psi} >= \begin{pmatrix} |\hat{\psi}^- > \\ \hat{\psi}^+ > \end{pmatrix}, \tag{3.27b}$$

where the upper symbol d represents isodual conjugation (anti-Hermitean conjugation), and $|\hat{\psi}^- >, \hat{\psi}^+ >$ represent the two-components wavefunctions for the isoelectron and the isopositron under the respective external fields of a proton and an antiproton considered as external. In different words, Eq. (3.24) can represents the Kronecker product of a neutron and an antineutron [52].

CASE II: IRREGULAR DIRAC-SANTILLI ISOEQUATIONS.
In this case we have the same equations (3.24) and the same isogammas (3.25), but the Pauli-Santilli isomatrices are irregular, e.g., are given by matrices (1.41) representing a mutation (isorenormalization) of the spin.

An important implication of the irregular equations is that the total angular momentum of the isoelectron is no longer conventionally quantized, and can assume, in particular, the null value,

$$J_e = s + j = \frac{1}{2}, \frac{3}{2}, ... \rightarrow \hat{I}_{\hat{e}} = 0. \tag{3.28}$$

as needed for an invariant representation of the neutron spin in its synthesis from a proton and an electron [38,39,52].

Note that in the above case we solely have the mutation of the *angular momentum,* but not that of spin, the latter being expected under energies dramatically bigger than those for the synthesis of the neutron.

3.6. Dirac's Generalization of Dirac's Equation. The identification of the background methods for the treatment of the isoscattering theory would be grossly deficient without an outline of the generalization of Dirac's equation achieved by Dirac himself in two of his last, vastly ignored papers [54,55] because admitting an essential isotopic structure as well as the null value of the total angular momentum so crucial for a quantitative representation of the neutron synthesis.

The *Dirac's generalization of Dirac equation* is given by [54]

$$(a_\mu \times \partial_\mu + \beta) \times q \times \psi = 0, \tag{3.29a}$$

$$q = \begin{pmatrix} q_1, p_1 \\ q_2, p_2 \end{pmatrix}, \quad \psi = \begin{pmatrix} psi_{1+}, \psi_{1-} \\ \psi_{2+}, \psi_{2-} \end{pmatrix}, \tag{3.29b}$$

where the reader can immediately recognize the role of the q-quantity as characterizing a right isomodular action which is at the foundation of the Lie-Santilli isotheory, as well as of Eq. (3.24)

By assuming

$$a_4 = I_{4x4}, \tag{3.30}$$

Dirac's a-matrices are characterized by the expression

$$a_\mu \times \beta \times a_\nu + a_\nu \times \beta \times a_\mu = 2 \times \beta \times \eta_{\mu\nu}, \tag{3.31}$$

where the reader will immediately see the same isotopic structure of iso-anticommutators (3.26), and $\eta_{\mu\nu}$ is the conventional Minkowski metric.

On the basis of the above structure, Dirac reached the following realization of the a- and β-matrices

$$\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad a_1 = i \times \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.32a}$$

$$a_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad a_3 = i \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (3.32b)$$

The total angular momentum is characterized by

$$S_{ij} = -(a_i \times \beta \times a_j - a_j \times \beta \times a_i) \times \frac{q \times q^t}{8}, \qquad (3.33)$$

where t stands for transposed, and possesses the eigenvalues

$$S^2 = S_{12}^2 + S_{23}^2 + S_{31}^2 =$$

$$= \frac{1}{8} \times (q_1^2 + p_1^2 + q_2^2 + p_2^2) = J \times (J+1), \qquad (3.34a)$$

$$J = \frac{1}{4} \times (q_1 + p_1 + q_2 + p_2) - \frac{1}{2} = \frac{1}{2} \times (n + n'), \qquad (3.34b)$$

$$n, \ n' \ = \ 0, 1, 2, 3, ... \qquad (3.34c)$$

thus *admitting the value $J = 0$ for the ground state.*

The historical aspect particularly significant for hadronic mechanics at large, and the isoscattering theory in particular, is that, without his knowledge, *Dirac's generalization of Dirac's equation possesses an irreducible isotopic structure* with isounit and isotopic element first identified in Ref. [13b]

$$\hat{I} = \beta^{-1}, \quad T = \beta, \qquad (3.35)$$

where the irreducibility is referred to the property that papers [52,53] become inconsistent unless *entirely* elaborated with respect to the isoproduct

$$A \hat{\times} B = A \times T \times B, \ T = \beta. \qquad (3.36)$$

We cannot close this section without the indication that, for structural consistency, Dirac's generalized equation cannot be formulated on the conventional Minkowski space $M(x, \eta, \mathcal{R})$ and must be formulated on the Minkowski-Santilli isospace $\hat{M}(\hat{x}, \hat{\eta}, \hat{\mathcal{R}})$, with isometric [13b]

$$\hat{\eta} = \beta \times \eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad (3, 37)$$

namely, Dirac not only had the intuition without elaboration of the isotopic formalism, but identified without his knowledge the first known *nondiagonal realization of the spacetime isometric* (3.37).

Rather than being an innocuous occurrence, the implications are far reaching because the line element now reads

$$x^{\hat{2}} = x^\mu \times (\beta_\mu^\rho \times \eta_{\rho\nu}) \times x^\nu = x^\mu \times \hat{\eta}_{\mu\nu} =$$
$$= x^1 \times x^3 - x^2 \times x^4 - x^3 \times x^1 - x^2 \times x^4 = -2 \times x^2 \times x^4, \qquad (3.38)$$

namely, when reformulated in an invariant way, *Dirac's isoequation (3.29) mutates spacetime from the conventional four-dimensions down to two-dimensions.*

The implications for the isoscattering theory are far reaching because, when the interior scattering region is characterized by Dirac's generalized equation, it loses completely the conventional four dimensions, by reducing spacetime solely to two dimensions, even though fully perceived from the outside as being four dimensional.

In summary, due to the power of his intuition, perhaps unprecedented in scientific history, Dirac should be consider the precursor of:

1) The isodual theory of antimatter, that originated from Dirac's negative unit $-I_{2\times 2}$ in the conventional γ_4 matrix [15];

2) The isotopic formalism and, consequently, the isoscattering theory, that originated from Dirac's generalized equation (3.29) [13b]; and

3) The first exact representation of the synthesis of the neutron from a proton and an electron in the core of a star requiring a null total angular momentum of the electron, that was first achieved by Dirac in Eq. (3.34) [39].

As a personal note, at one of his last participations at scientific meetings in the early 1980s in Florida, Santilli briefly presented to Dirac the isotopic formalism with the connection to his papers [54,55] and the indication of the strong convergence of conventionally divergent series under the isotropic product $A \times T \times B - B \times T \times A, |T| \ll 1$, thus including the possible achievement of Dirac's dream for a scattering theory without divergences.

Contrary to his normally reserved nature, Dirac showed great interest. Unfortunately, Dirac's death in 1984 deprived the then newly born isotopies of his most powerful supporter, thus delaying collegial acceptance for decades. A rewarding aspect is that, with the power of his intuition, Dirac did in fact see prior to his death the technical feasibility of his dream of a scattering theory without divergencies hereon called the *Dirac's problem* (for more details in the episode, one may consult Ref. [16d]).

3.7. Experimental, Verifications. The presentation of the isoscattering theory in Paper III could be misleading to non-expert in the field without the indication of rather significant experimental verifications and novel applications of hadronic mechanics in various quantitative sciences along the following setting:

I) CONDITIONS OF EXACT VALIDITY OF QUANTUM MECHANICS. They are assumed as being those of special relativity, namely, the conditions of *exterior dynamical. problems* (point-particles and electromagnetic waves propagating in vacuum), since the latter do admit the effective point-like abstraction of particles necessary for the validity of the local-differential and topological, foundations of the theory. Therefore, these papers assume that quantum mechanics is *exactly* valid for the structure of the hydrogen atom, particles in accelerators, crystals, and numerous other exterior problems.

II) CONDITIONS OF APPROXIMATE VALIDITY OF QUANTUM MECHANICS. These papers assume that quantum mechanics is only *approximately* valid for *interior dynamical problems* (extended particles and electromagnetic waves moving within physica;l media) due to expected nonlinear, nonlocal and nonpotential interactions compared to the strictly linear, local and potential character of the mechanics. In particular, these papers assume that all arbitrary parameters and functions of unknown physical origin whose values are fitted from experimental data, represent in reality *deviations* from the very axioms of quantum mechanics, as illustrated by the following representative cases:

II-A) The inability by quantum mechanics to achieve an exact representation of the binding energy and other features of the hydrogen molecule, that required the use of the so-called "screened Coulomb potentials" of the type $V^*(r) = f(r) \times e^2/r$ with evident loss of quantized levels and other problematic aspects. By comparison, the use of hadronic mechanics has permitted the achievement of a numerically exact and invariant representation of the binding energy of the hydrogen and other molecules from first axiomatic principles without *ad hoc* functions of unknown physical origin or meaning. The representation is achieved via the sole admission of non-Hamiltonian interactions originating from the deep mutual penetration of the wavepackets of valence electrons, and provides convergent power series dramatically faster than those of quantum mechanics [14]. Similar insufficiencies of quantum mechanics exist for the representation of the simplest possible nucleus, the deuteron, not to mention very large deviations for heavy nuclei, for which the use of hadronic mechanics has provided distinct advances [16d,27].

II-B) The representation via relativistic quantum mechanics of the two-point functions of the Bose-Einstein correlation in a way compatible with experimental data requires *four* arbitrary parameters of unknown physical origin or meaning (called the "chaoticity parameters"), that are fitted from experimental data and relativistic quantum mechanics is then claimed as being valid. However, the vacuum expectation values of a diagonal two-dimensional Hamiltonian can at best allow *two* arbitrary parameters, thus indicating that the very structure of the Bose-Einstein correlation is beyond the representational capabilities of relativistic quantum mechanics. After all, said correlation originates from the deep mutual penetration and consequential annihilation protons and antiprotons under which linear, local and potential treatments can at best be approximately valid. By comparison, relativistic hadronic mechanics achieves an exact and invariant representation of the Bose-Einstein correlation at both high and low energies via the four characteristic quantities of isospacetime, three of which represent the very elongated fireball and the forth represents its density [16d,27].

II-C) As it is equally well known, the representation of the behavior of the meanlives of unstable hadrons with speed is equally achieved via the introduction of a number of parameters, functions, expansions, cut off and other mechanisms to achieve compatibility with relativistic quantum mechanics. However, the latter theory is fully reversible over time, while spontaneous hadrons decays are strictly irreversible. Hence, the latter theory cannot possibly be exact for the former events. The use of the Lie-admissible branch of hadronic mechanics permits an exact representation of the *irreversible character* of the decay in a way compatible with experimental. data from first axiomatic principles without conceptual or axiomatic adaptations [16d,27].

III) CONDITIONS OF INAPPLICABILITY OF QUANTUM MECHANICS. Throughout the 20th century, it was widely believed that quantum mechanics can be applied to all possible conditions of particles existing in the universe. At a meeting in February 1978, Herman Feshback (then from MIT) confirmed to Santilli (then from Harvard) that Schrödinger's equation is inapplicable for the synthesis of the neutron from a proton and an electron as occurring in the core of stars, $p^+ + e^- \rightarrow n + \nu$, because the rest energy of the neutron is $0.782 MeV$ *bigger* than the sum of the rest energies of the proton and the electron, thus requiring a *positive binding energy* under which the indicial equation no longer admits physically acceptable solution. The origin of the missing energy from a possible relative kinetic energy between the proton and the electron has to be excluded due to their extremely small cross section at the indicated energy, and the same holds for the antineutrino in the conjugate reaction $p^+ + e^- + \bar{\nu} \rightarrow n$, thus establishing the *inapplicability* (and certainly not the violation) of quantum mechanics s for a quantitative treatment of the problem considered. At the same meeting, that signals the birth of hadronic mechanics, Santilli indicated to Feshback that a nonunitary generalization of quantum mechanics in general, and of Schrödinger's equation in particular, does indeed admit physically meaningful for hadron syntheses with "missing energies" solutions for various reasons, e.g., the anomalous renormalization of the rest energy of the electron due to non-Hamiltonian interactions under which the binding energy returns to acquire the conventional *negative* Coulomb form between opposite charges. The original proposal [4,5] published in April 1978 to construct hadronic mechanics provides a complete solution for the synthesis also requiring a "positive binding energy" $e^+ + e^- \rightarrow \pi^o$, while the full solution at the nonrelativistic and relativistic levels of the synthesis of the neutron was achieved after decades of study of the isotopies of Lie's theory and the discovery of the irregular isotopies of the spin symmetry outlined in Section 1 (see Ref. [52] for a general review).

We ended Paper I with the indication that, despite one millennium of studies, our knowledge of light is still at its infancy. It is appropriate to end this paper with the indication that quantum mechanics has been a beautiful episode in the history of physics but, in view of its evident limitations in face of the complexities of the physical world, any belief of its final character for all of nature causes the existing from the boundaries of science, the sole scientific issue, for which these papers have been written, being the selection via experiments, rather than personal or conceptual indications, of its appropriate structural generalization.

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References

[1] R. M. Santilli, "Nonunitary-Isounitary Scattering Theory of Hadronic mechanics, I: Conceptual and Mathematical. Foundations, " submitted for publication,

[2] H. A. Lorentz, *Versuch einer Theorie der Elektrishen und Magnetishen Erscheinungen in Bewenglen Korpern*, Leyda (1895).

[3] R. M. Santilli, Nuovo Cimento **51**, 570 (1967), available in free download from the link
http://www.santilli-foundation.org/docs/Santilli-54.pdf

[4] R. M. Santilli, "On a possible Lie-admissible covering of Galilei's relativity in Newtonian mechanics for nonconservative and Galilei form-noninvariant systems," Hadronic J. **1**, 223-423 (1978), available in free pdf download from
http://www.santilli-foundation.org/docs/Santilli-58.pdf

[5] R. M. Santilli, "Need of subjecting to an experimental verification the validity within a hadron of Einstein special relativity and Pauli exclusion principle," Hadronic J. **1**, 574-901 (1978), available in free pdf download from
http://www.santilli-foundation.org/docs/Santilli-73.pdf

[6] R. M. Santilli, Algebras, Groups and Geometries **10**, 273 (1993), available as free downlaod from
http://www.santilli-foundation.org/docs/Santilli-34.pdf

[7] R. M. Santilli, Rendiconti Circolo Matematico Palermo, Suppl. **42**, 7-82 (1996), available as free download from
http://www.santilli-foundation.org/docs/Santilli-37.pdf

[8] R. M. Santilli, Found. Phys. **27**, 1159 (1997), available in free pdf download from the link
http://www.santilli-foundation.org/docs/Santilli-06.pdf

[9] R. M. Santilli, "Lie-admissible invariant representation of irreversibility for matter and antimatter at the classical and operator levels," Nuovo Cimento B bf 121, 443 (2006), available in free pdf download from the link
http://www.santilli-foundation.org/docs/Lie-admiss-NCB-I.pdf

[10] R. M. Santilli, *Foundation of Theoretical Mechanics,* Volume I (1978) [10a], and Volume II (1982) [10b],Springer-Verlag, Heidelberg, Germany, available as free download from
http://www.santilli-foundation.org/docs/Santilli-209.pdf
http://www.santilli-foundation.org/docs/santilli-69.pdf

[11] R. M. Santilli, *Lie-Admissible Approach to the Hadronic Structure,* Vol. I (1978) [11a] and Vol. II (1982) [11b], Hadronic Press, availabl;e as free downloads from the links
http://www.santilli-foundation.org/docs/Santilli-71.pdf
http://www.santilli-foundation.org/docs/Santilli-72.pdf

[12] R. M. Santilli, *Isotopic Generalizations of Galilei and Einstein Relativities,* Vol. I (1991) [12a] and Vol. II (1991) [12b], Hadronic Press, Palm Harbor, Florida, available in free pdf download from
http://www.santilli-foundation.org/docs/Santilli-01.pdf
http://www.santilli-foundation.org/docs/Santilli-35.pdf

[13] R. M. Santilli, *Elements of Hadronic Mechanics*, Vol. I (1995) [13a], Vol. II 91995) [13b] Academy of Sciences, Kiev, available in free pdf downloads from
http://www.santilli-foundation.org/docs/Santilli-300.pdf
http://www.santilli-foundation.org/docs/Santilli-301.pdf

[14] R. M. Santilli, *Foundations of Hadronic Chemistry, with Applications to New Clean nergies and Fuels,* Kluwer Academic Publishers (2001), available as free download from
http://www.santilli-foundation.org/docs/Santilli-113.pdf

[15] R. M. Santilli, *Isodual Theory of Antimatter with Applications to Antigravity, Grand Unifications and Cosmology,* Springer (2006).

[16] R. M. Santilli, *Hadronic Mathematics, Mechanics and Chemistry,*, Vol. I [16a], II [16b], III [16c], IV [16d] and [16e], international academnioc press, (2008), available as free downlaods from http://www.i-b-r.org/Hadronic-Mechanics.htm

[17] H. C. Myung,, *Lie Algebras and Flexible Lie-Admissible Algebras,* Hadronic Press (1982), available as free download from http://www.santilli-foundation.org/docs/Santilli-107.pdf

[18] H, C, Myung, *Malcev-Admissible Algebras,* H. C. Myung, Birkhauser (1986).

[19] A. K. Aringazin, A. Jannussis, F. Lopez, M. Nishioka and B. Veljanosky, *Santilli's Lie-Isotopic Generalization of Galilei and Einstein Relativities,* Kostakaris Publishers, Athens, Greece (1991), available as free download from http://www.santilli-foundation.org/docs/Santilli-108.pdf

[20] D. S. Sourlas and G. T. Tsagas, *Mathematical Foundation of the Lie-Santilli Theory,* Ukraine Academy of Sciences 91993), available as free download from http://www.santilli-foundation.org/docs/santilli-70.pdf

[21] A. Elduque and H. C, Myung, *Mutations of Alternative Algebras,* Kluwer academic Publishers(1994).

[22] J. Lôhmus, E. Paal, and L. Sorgsepp, *Nonassociative Algebras in Physics* (Hadronic Press, Palm Harbor, 1994), available as free download from http://www.santilli-foundation.org/docs/Lohmus.pdf

[23] J. V. Kadeisvili, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities,* Ukraine Academy of Sciences, Second edition (1997), available as free download from http://www.santilli-foundation.org/docs/Santilli-60.pdf

[24] Chun-Xuan Jiang, *Foundations of Santilli Isonumber Theory,* International Academic Press (2001), available as free download from http://www.i-b-r.org/docs/jiang.pdf

[25] Raul M. Falcon Ganfornina and Juan Nunez Valdes, *Fundamentos de la Isoteoria de Lie-Santilli,* International Academic Press (2001), available as free downlaod from
http://www.i-b-r.org/docs/spanish.pdf

[26] B. Davvaz, *Hyperings Theory and Applications,* International Academic Press (2007), available as free downlaod from
http://www.santilli-foundation.org/docs/Davvaz.pdf

[27] A. O. E. Animalu, M. Cloonan and D. Gandzha, *New Sciences for a New Era" Mathematical, Physical and Chemical Discoveries of Ruggero Maria Santilli,* in preparation, preli,kinary versionm available in the website
http://www.santilli-foundation.org/santilli-scientific-discoveries.html

[28] J. V. Kadeisvili, "An introduction to the Lie-Santilli isotopic theory," Mathematical Methods in Applied Sciences **19**, 1349 (1996), available in the website
http://www.santilli-foundation.org/docs/Santilli-30.pdf

[29] Raul M. Falcon Ganfornina and Juan Nunez Valdes, "Studies on the Tsagas-Sourlas-Santil;li Isotopology," Algebras, Groups and Geometries **20**, 1 (2003), available from the link
http://www.santilli-foundation.org/docs/isotopologia.pdf

[30] G. Tsagas, "Studies on the classification of Lie-Santilli algebras," Algebras, Groups and Geometries **13**, 129- (1996). available from the link
http://www.santilli-foundation.org/docs/Tsagas.pdf

[31] R. M. Santilli, "Isotopies of Lie symmetries, I" Hadronic J. **8**, 36 (1985), available as free download from
http://www.santilli-foundation.org/docs/santilli-65.pdf

[32] R. M. Santilli, "Isotopies of Lie symmetries, II" Hadronic J. **8**, 85 (1985), available as free download from
http://www.santilli-foundation.org/docs/santilli-65.pdf

[33] R. M. Santilli, JINR rapid Comm. **6**. 24-38 (1993), available as free downlaod from
http://www.santilli-foundation.org/docs/Santilli-19.pdf

[34] R. M. Santilli, Acta Applicandae Mathematicae **50**, 177 (1998), available as free download from
http://www.santilli-foundation.org/docs/Santilli-19.pdf

[35] R. M. Santilli, Lettere Nuovo Cimento **37** 545 (1983),, available as free downlaod from
http://www.santilli-foundation.org/docs/Santilli-50.pdf

[36] R. M. Santilli, Lettere Nuovo Cimento **38**, 509 (1983), available as free downlaod from
http://www.santilli-foundation.org/docs/Santilli-51.pdf

[37] R. M. Santilli, Moscow Phys. Soc. **3**, 255 (1993), available as free download from
http://www.santilli-foundation.org/docs/Santilli-40.pdf

[38] R. M. Santilli, "Recent theoretical and experimental evidence on the synthesis of the neutron," Communication of the Joint Institute for Nuclear Research, Dubna, Russia, No. E4-93-252 (1993), available as free download from
http://www.santilli-foundation.org/docs/Santilli-18.pdf

[39] R. M. Santilli, Chinese J. System Engineering and Electronics **6**, 177 (1995, available as free download from
http://www.santilli-foundation.org/docs/Santilli-18.pdf

[40] R. M. Santilli, Intern. J. Modern Phys. D **7**, 351 (1998), available as free download from
http://www.santilli-foundation.org/docs/Santilli-35.pdf

[41] J. V. Kadeisvili, "Direct universality of the Lorentz-Poincare'-Santilli isosymmetry for extended-deformable particles, arbitrary speeds of light and all possible spacetimes" in *Photons: Old problems in Light of New Ideas,* V. V. Dvoeglazov Editor Nova Science (2000, available

as free download from
http://www.santilli-foundation.org/docs/Santilli-25.pdf

[42] A. K. Aringazin and K. M. Aringazin, "Universality of Santilli's iso-Minkowskian geometry" in *Frontiers of Fundamental Physics,* M. Barone and F. Selleri, Editors Plenum 91995), available as free download from
http://www.santilli-foundation.org/docs/Santilli-29.pdf

[43] R. M. Santilli, "Direct universality of isospecial relativity for photons with arbitrary speeds" in *Photons: Old problems in Light of New Ideas,* V. V. Dvoeglazov Editor Nova Science (2000), available for free downlaod from
http://www.santilli-foundation.org/docs/Santilli-24.pdf

[44] R. M. Santilli, "Isotopic quantization of gravity and its universal isopoincare' symmetry" in the *Proceedings of "The Seventh Marcel Grossmann Meeting in Gravitation,* SLAC 1992, R. T. Jantzen, G. M. Keiser and R. Ruffini, Editors, World Scientific Publishers pages 500-505(1994), available in free download from
http://www.santilli-foundation.orgdocs/Santilli-120.pdf

[45] R. M. Santilli, "Unification of gravitation and electroweak interactions" in the proceedings of the *Proceedings of the Eight Marcel Grossmann Meeting in Gravitation,* Israel 1997, T. Piran and R. Ruffini, Editors, World Scientific, pages 473-475 (1999), available as free download from
http://www.santilli-foundation.org/docs/Santilli-137.pdf

[46] R. M. Santilli, "Isotopic grand unification with the inclusion of gravity," Found. Phys. Letters **10**, 307 (1997), available as free download from
http://www.santilli-foundation.org/docs/Santilli-04.pdf

[47] R. M. Santilli, "Experimental verifications of isoredshift with expected lack of "big bang," "dark matter," and "dark energy," available as free download from

http://www.santilli-foundation.org/docs/Isoredshift-Paper-01-11-10.pdf

[48] R. M. Santilli, Galilean Electrodynamics, Summer 2006, **17**, 43 (2006), available as free download from http://www.i-b-r.org/Incons.GravFinalGED-I.pdf

[49] A. O. E. Animalu and R. M. Santilli, in *Hadronic Mechanics and Nonpotential Interactions,* M. Mijatovic, Editor, Nova Science, New York, pp. 19 (l990).

[50] H. C. Myung and R. M. Santilli, Hadronic Journal Vol. 5, pages 1277-1366 (1982). available as free download from http://www.santilli-foundation.org/docs/Santilli-201.pdf

[51] R. Mignani, Hadronic J. **4**, 2185 (1981).

[52] J. V. Kadeisvilki,. "The Rutherford-Santilli Neutron," Hadronic J. **31**, 1 (2008), available in free pdf download from the link
http://www.i-b-r.org/Rutherford-Santilli-II.pdf
also available in htlm format ffrom the website
http://www.i-b-r.org/Rutherford-Santilli-neutron.htm

[53] R. M. Santilli, "Relativistic hadronic mechanics: nonunitary, axiom-preserving completion of relativistic quantum mechanics," Found. Phys. Vol. **27**, 625 (1997), available as free download from http://www.santilli-foundation.org/docs/Santilli-15.pdf

[54] P. A. M. Dirac, Proc. Royal Soc. A **322**, 435 (1971)

[55] P. A. M. Dirac, Proc. Royal Soc. A **328**, 1 (1972).