

## DYNAMICAL SYSTEMS AND JORDAN STRUCTURES

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**Abstract:** I shall firstly give a small introduction to Jordan structures and their applications. Then I shall emphasize the Romanian contributions to this field of research. Next I shall point out recent applications of Jordan algebras to dynamical systems, and I shall suggest some open problems.

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### 1. Jordan Structures and their Applications

There exist three kinds of *Jordan structures*, namely, *algebras*, *triple systems*, and *pairs*. Since the creation of Jordan algebras (at the beginning of '30) by Pasqual Jordan – a quantum mechanics german specialist – see Jordan [38], [39], [40], and denominated with his name in 1946 by A.A. Albert, an improvement of the mathematical foundation of quantum mechanics was made, but the problem was not definitively solved. Fifty years later (in '80), another tentative was made using a kind of mixed structures, namely Jordan-Banach algebras, but the problem of the mathematical foundation of quantum mechanics is still an open problem!

Anyway, in the meantime, Jordan structures have been intensively studied by mathematicians, and a big number of important results has been obtained. At the same time, an impressive variety of applications have been explored with several surprising connections. The study of Jordan structures and their applications is at present a wide-ranging field of mathematical research.

The first book on Jordan algebras is the excellent monograph “Jordan-Algebren” written by the late Professors Dr. Hel Braun (Hamburg) and Max Koecher (München), and published by Springer-Verlag in 1966 (see [6]). We must add here the important monographs (or papers) written by Chu [7], Helwig [24], Jacobson [37], Koecher [52, 53], Loos [66, 67, 68], Meyberg [71, 72], Neher [74], Springer and Veldkamp [79], Zelmanov [89, 90], Zhevlakov, Slinko, Shestakov, Shirshov [91] (for a more detailed information, see Iordănescu [36]).

**Definition 1.1.** Let  $\mathcal{A}$  be a vector space over a field  $\mathbb{F}$  with characteristic not two. Let  $\varphi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  be an  $\mathbb{F}$ -bilinear map, denoted  $\varphi(x, y) := xy$ , satisfying

$$xy = yx, \quad x^2(yx) = (x^2y)x \quad \text{for all } x, y \in \mathcal{A}.$$

Then  $\mathcal{A}$  together with the product defined by  $\varphi$  is called a (linear) *Jordan algebra over  $\mathbb{F}$* .

**Example 1.1.** If  $A$  is an associative algebra over a field  $\mathbb{F}$  with characteristic not two and

$$\varphi(x, y) := \frac{x \cdot y + y \cdot x}{2},$$

where the dot denotes the associative product in  $A$ , then  $A$  becomes a (linear) Jordan algebra over  $\mathbb{F}$ , usually denoted by  $A^{(+)}$ .

**Definition 1.2.** A real Jordan algebra is called *formally real* if for any two its elements  $x$  and  $y$ ,  $x^2 + y^2$  implies  $x = y = 0$ .

**Notation.** On a Jordan algebra  $\mathcal{A}$  consider the *left multiplication*  $L$  given by  $L(x)y := xy$ ,  $x, y \in \mathcal{A}$ .

**Remark 1.1.** In general,  $L(x, y) \neq L(x)L(y)$ .

**Definition 1.3.** The map  $P$  defined by  $P(x) := 2L^2(x) - L(x^2)$  for all  $x \in \mathcal{A}$  is called the *quadratic representation of  $\mathcal{A}$* .

**Example 1.2.** When  $\mathcal{A} = A^{(+)}$ , then  $P(x)y = x \cdot y \cdot x$ .

**Proposition 1.1.** For any  $x, y \in \mathcal{A}$  we have the following fundamental formula

$$P(P(x)y) = P(x)P(y)P(x).$$

**Proposition 1.2.** Suppose that  $\mathcal{A}$  has the identity element  $e$  and  $x \in \mathcal{A}$ ; then  $P(x)$  is an automorphism of  $\mathcal{A}$  if and only if  $x^2 = e$ . In this situation  $P$  is involutive.

**Definition 1.4.** An element  $x \in \mathcal{A}$  is called *invertible* if  $P(x)$  is bijective. In this case the *inverse* of  $x$  is given by  $x^{-1} := (P(x))^{-1}(x)$ .

**Example 1.3.** If  $\mathcal{A} = A^{(+)}$  then  $x^{-1}$  is the usual inverse of  $x$ .

**Remark 1.2.** We have  $(P(x))^{-1} = P(x^{-1})$ .

**Remark 1.3.**  $x \in \mathcal{A}$  is invertible with the inverse if and only if  $xy = e$  and  $x^2y = x$ .

**Definition 1.5.** Let  $f \in \mathcal{A}$  and define a new product by

$$x \perp y := x(yf) + y(xf) - (xy)f.$$

The vector space  $\mathcal{A}$  together with  $\perp$  is called the *mutation* (or *homopote*) of  $\mathcal{A}$  with respect to  $f$  and it is denoted by  $\mathcal{A}_f$ .

**Example 1.4.** If  $\mathcal{A} = A^{(+)}$  then  $\mathcal{A}_f = (A_f)^{(+)}$ , where  $A_f$  is the associative mutation of  $A$  (i.e.,  $x \circ_f y := x \cdot f \cdot y$ ).

**Proposition 1.3.** Any  $\mathcal{A}_f$  of  $\mathcal{A}$  is a Jordan algebra and  $P_f(x) = P(x)P(f)$ .

**Proposition 1.4.**  $\mathcal{A}_f$  has unit element if  $f$  is invertible in  $\mathcal{A}$ . In this case it is  $f^{-1}$ .

**Remark 1.4.** If  $f^{-1}$  exists then  $\text{Inv } \mathcal{A} = \text{Inv } \mathcal{A}_f$ , where by  $\text{Inv } \mathcal{A}$  we have denoted the set of invertible elements of  $\mathcal{A}$ .

**Proposition 1.5.** The set  $\Gamma(\mathcal{A})$  of linear bijective maps  $W$  of  $\mathcal{A}$  for which there exist linear bijective maps  $W^*$  on  $\mathcal{A}$  such that  $P(Wx) = WP(x)W^*$  for all  $x \in \mathcal{A}$  is an algebraic group.

**Comment.** The notation  $W^*$  is justified by the fact that if  $\mathcal{A}$  is *real semi-simple* and  $\lambda$  is its *trace form* (i.e.,  $\lambda(x, y) := \text{Tr } L(xy)$ ), then for  $W \in \Gamma(\mathcal{A})$ ,  $W^*$  is the adjoint of  $W$  with respect to  $\lambda$ .

**Definition 1.6.**  $\Gamma(\mathcal{A})$  from Proposition 1.5 is called the *structure group* of  $\mathcal{A}$ .

**Remark 1.5.** From Proposition 1.1 it follows that  $P(x) \in \Gamma(\mathcal{A})$  if and only if  $x$  is invertible.

**Remark 1.6.**  $\text{Aut } \mathcal{A} = \{W \mid W\Gamma(\mathcal{A}), We = e\} \subset \Gamma(\mathcal{A})$ .

**Theorem 1.1.** Every formally real finite-dimensional Jordan algebra is a direct sum of the following algebras  $H_p(\mathbb{R})^{(+)}$ ,  $H_p(\mathbb{C})^{(+)}$ ,  $H_p(\mathbb{H})^{(+)}$ ,  $H_3(\mathbb{O})^{(+)}$ ,  $J(B)$ , where  $H_p(\mathbb{F})^{(+)}$  denotes the algebra of Hermitian  $p \times p$ -matrices over  $\mathbb{F}$  ( $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  or  $\mathbb{O}$ ) with the product  $xy := \frac{1}{2}(x \cdot y + y \cdot x)$ , and  $J(B) = \mathbb{R}1 \oplus X$ , where  $B$  is a real-valued symmetric bilinear positive defined form on the real vector space  $X$  equipped with the product

$$(\lambda, u)(\mu, v) := (\lambda\mu + B(u, v), \lambda v + \mu u).$$

**Proposition 1.6.** *A Jordan algebra is formally real if and only if its trace form is positive defined.*

**Notation.** Suppose that  $\mathcal{A}$  has a unit element  $e$ . Then we set

$$x^0 := e, \quad \exp x := \sum_{n \geq 0} \frac{x^n}{n!}, \quad \text{and} \quad \exp \mathcal{A} := \{\exp x \mid x \in \mathcal{A}\}.$$

**Proposition 1.7.** *If  $\mathcal{A}$  is a formally real Jordan algebra then it possesses a unit element and*

$$\exp \mathcal{A} = \{x^2 \mid x \text{ invertible in } \mathcal{A}\}.$$

**Remark 1.7.** If we suppose  $\mathcal{A}$  from Proposition 1.7 endowed with the natural topology of  $\mathbb{R}^n$ , then  $\text{Inv}_0(\mathcal{A}) = \exp \mathcal{A}$  and the map  $x \rightarrow \exp x$  is bijective. By  $\text{Inv}_0(\mathcal{A})$  we have denoted the connected component of  $\text{Inv}(\mathcal{A})$  containing the identity.

**Definition 1.7.** A Jordan algebra  $\mathfrak{a}$  over a field  $\mathbb{F}$  is called *central simple* if it is central and simple, i.e., it has a unit element  $e$ , its centre

$$\{c \mid c \in \mathfrak{a}, (ca)b = c(ab), (ac)b = a(cb) \text{ for all } a, b \in \mathfrak{a}\}$$

coincides with  $\mathbb{F}e$  and it has no proper ideals.

The classification of *simple* Jordan algebras can be reduced to that of *central simple* algebras, and for these A.A. Albert proved the existence of a finite-dimensional extension field  $\mathcal{F}$  of the underlying field such that the scalar extensions  $\mathcal{A}_{\mathcal{F}}$  are contained in the following list of *split algebras*:

- A. the algebra  $M_n(\mathcal{F})^{(+)}$  of  $n \times n$ -matrices over  $\mathcal{F}$ ;
- B. the subalgebra of  $M_n(\mathcal{F})^{(+)}$  of symmetric matrices;
- C. the subalgebra of  $M_{2m}(\mathcal{F})^{(+)}$  of symplectic matrices, i.e., of matrices that are symmetric with respect to the involution  $x \rightarrow q^{-1}x'q$ , where  $x'$  denotes the transposed of  $x$ , and  $q = \begin{pmatrix} 0 & \text{Id}_m \\ -\text{Id}_m & 0 \end{pmatrix}$ ,  $\text{Id}_m$  being the  $m \times m$ -identity matrix;
- D. the algebra with basis  $\{e_0, e_1, \dots, e_n\}$  and the multiplication table  $e_0e_i = e_i$ ,  $e_i^2 = e_0$ ,  $e_ie_j = 0$  ( $i \neq j$ );
- E. the algebra of Hermitian  $3 \times 3$ -matrices with entries in an octonion algebra relative to the multiplication  $xy := \frac{1}{2}(x \cdot y + y \cdot x)$ .

If in a given Jordan algebra we define a triple product by

$$(xyz) := (xy)z + (zy)x - y(xz),$$

then it satisfies the following two identities

$$(xyz) = (zyx), \tag{JTS1}$$

$$(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz)). \tag{JTS2}$$

In general, a module with a trilinear composition  $\{xyz\}$  satisfying (JTS1) and (JTS2) is called a *Jordan triple system*.

**Theorem 1.2.** *If  $T$  is a Jordan triple system and  $a$  an element of  $T$ , then  $T$  together with the product  $(x, y) \rightarrow \frac{1}{2}\{xay\}$  becomes a Jordan algebra, denoted  $T_a$ . Conversely, a Jordan algebra induces a Jordan triple system in the same vector space by setting  $\{xyz\} := P(x, z)y$ , where  $P(x, z) := 2(L(x)L(z) + L(z)L(x) - L(xz))$ .*

**Definition 1.8.** Let  $\mathbb{K}$  be a unital commutative ring such that 2 is invertible in  $\mathbb{K}$ . Assume all  $\mathbb{K}$ -modules to be unital and to possess no 3-torsion (i.e., no nonzero elements  $x$  such that  $3x = 0$ ). A pair  $V = (V^+, V^-)$  of  $\mathbb{K}$ -modules endowed with two trilinear maps  $V^\sigma \times V^{-\sigma} \times V^\sigma \rightarrow V^\sigma$ , written as  $(x, y, z) \rightarrow \{xyz\}_\sigma$ ,  $\sigma = \pm$ , satisfying

$$\{xyz\}_\sigma = \{zyx\}_\sigma,$$

$$\{xy\{uvz\}_\sigma\}_\sigma - \{uv\{xyz\}_\sigma\}_\sigma = \{\{xyu\}_\sigma\}vz\}_\sigma - \{u\{yxv\}_{-\sigma}z\}_\sigma$$

for  $\sigma = \pm$  is called a (linear) *Jordan pair over  $\mathbb{K}$* .

**Remark 1.8.** Jordan algebras can be regarded as a generalization of *symmetric* matrices, while the (linear) Jordan structures (triple systems or pairs) can be regarded as a generalization of *rectangular* matrices.

**Remark 1.9.** In 1966 Kevin McCrimmon extended the theory of linear Jordan algebras to the case of an arbitrary commutative unital underlying ring by defining unital *quadratic* Jordan algebras. Kurt Meyberg defined in 1972 *quadratic* Jordan triple systems, and two year later, in 1974, Ottmar Loos defined *quadratic* Jordan pairs. In what follows we shall refer only to the linear theory.

The first important application of Jordan algebras to (projective) geometry was made in 1945 by Pasqual Jordan himself, but published only four years later (in 1949) – as (yet another one!) consequence of the second world war! (see Jordan [41]). Forty years ago (in 1965) appeared the important paper “Über Jordan-Algebren und kompakte Riemannsche symmetrischen Räume von Rang 1” by Ulrich Hirzebruch [27], where the description of Riemannian symmetric spaces of rank 1 (essential entities from differential geometry) in terms of

Jordan algebras is given. Five years later (in 1970) the late Professor Dr. Karl-Heinz Helwig (München, Germany) has extended in [25] the Jordan algebra descriptions of geometrical entities given previously by Hirzebruch in [27]. In 1975, Erhard Neher detailed in his Diplomarbeit (University of Münster, Germany) entitled “Differentialgeometrische Aspekte von Idempotenten in reellen Jordan-Algebren” the Hirzebruch’s study.

Let us mention here the list of the prominent examples of the applications of Jordan structures to geometry and analysis given by Harald Upmeyer in [83]: “a) Characterization of domains of positivity (self-dual homogeneous convex cones) in terms of formally real Jordan algebras; b) Coordinatization theorems in projective geometry over rings (octonion planes, Barbilian structures); c) Facially homogeneous cones in real Hilbert spaces and Jordan-Hilbert algebras; d) Affine-geometric description of quantum mechanical state spaces in terms of dual Banach Jordan algebras; e) Jordan theoretic characterization of infinite-dimensional symmetric Banach manifolds (bounded Hermitian symmetric domains and their compact-type duals); f) Isoparametric hypersurfaces in spheres and Jordan-Peirce decomposition, functorial approach to Jordan theoretic differential geometry; g) Harmonic analysis on symmetric cones, holomorphic discrete series representations of semisimple Lie groups, Hua operators and harmonic functions; h) Contractive projections on Jordan triples and relations to noncommutative functional analysis and operator spaces; i) Toeplitz operators, Berezin transform and Jordan theoretic description of holomorphic boundary orbits; j) Real symmetric domains, their representations and quantization procedures.” There must be added the recent significant applications to Cauchy-Riemann geometry given by Wilhelm Kaup in cooperation with Dmitri Zaitsev (see Kaup and Zaitsev [49, 50, 51]).

We must add here the monographs (or papers) written by Benz [4], Bertram [5], Chu [8, 9], Chu and Isidro [7], Chu and Lau [10], Chu and Mellon [11], Dorfmeister [12], Dorfmeister and Neher [13, 14, 15], Emch [16], Faulkner [17, 18, 19], Ferus, Karcher, and Mnzner [20], Guz [21, 22], Hanche-Olsen and Størmer [23], Kaup [42, 43, 44, 45, 46, 47], Kaup and Upmeyer [48], Leissner [57, 58, 59, 60], Loos [69], Moufang [73], Rodriguez-Palacios [77], Springer and Weldkamp [79], Upmeyer [81, 82], Veldkamp [84], Walcher [87].

Besides all the above mentioned applications of Jordan structures inside mathematics, we must point out other very interesting and – sometimes – surprising applications of these structures to physics (more precisely, to soliton theory, to string theory, to quantum gravity, and to supersymmetry), to genetics (more precisely, to population genetics – see, for instance, the monograph “Algebras in Genetics” by Angelika Wrz-Busekros [88]), to statistics, to biology

(more precisely, to the theory of color perception), to dynamical systems (for more detailed information see Iordanescu [34, 36]).

As it is easily to find out, the contributions of the German school of mathematics to the theory of Jordan structures and their applications are fundamental.

## **2. Romanian Contributions to the Study of Jordan Structures and their Applications**

The interest in Jordan structures began in Romania in 1965 with the papers [30, 31, 32] by the Romanian physicist Dumitru B. Ion. Since 1966, independently of the German school, Iulian Popovici, Adriana Turtoi and myself, following a suggestion of our professor Gheorghe Vranceanu [85, 86], have studied the spaces (Riemannian or pseudo-Riemannian) associated with various kinds of real Jordan algebras. Our results have been presented in various papers, and afterwards systematized in the book [75].

In 1979, impressed by the deep results obtained by the late Professor Dr. Hel Braun and the late Professor Dr. Max Koecher, as well as by their former students, I wrote the monograph [28] which had a significant impact and is mentioned in almost all important reference books and papers published after 1980, and also in “Encyclopaedia of Mathematical Sciences”, **57**, Springer-Verlag, Berlin, 1995, on pp. 243 and 277. As a consequence of the very rich material (reprints, Ph.D. Thesis, etc.) that I have received after [33] was sent over the world, and the progresses made in the meantime in the field, I wrote – in 1990 – its extended version [34].

The recent book [1] by Asadurian and Stefanescu is devoted to the structure and representations of Jordan algebras. The appearance of their book, as well as the Ph.D. Thesis of Asadurian in January 2000, were determined by my monographs [33, 34].

In 2000, as a result of a Spanish-Romanian project (in the frame of the Scientifical and Technical Cooperation between the Spanish and the Romanian Governments, Professor Dr. Jose M. Isidro from the University of Santiago de Compostela being my Spanish partner), I gave the final form of another monograph, namely [35].

An up-dated and extended version of [35], the most recent applications being emphasized, is my recent monograph [36]. Let us mention here the words of Guy Roos: “This book provides an impressive broad survey of the use of Jordan structures (Jordan algebras and Jordan triples) in other branches of mathematics, especially in geometry and physics. It contains “only” definitions,

results and comments, including open problems and many historical comments. For details and proofs, the reader is referred to an extensive bibliography of 847 items" ... (see Roos [78]).

Firstly, the important construction of Koecher [54], as well as some classical results, must be recalled.

Let  $\mathcal{A}$  be a *real* Jordan algebra,  $\dim \mathcal{A} = n$ , and with unit element  $e$ .

**Remark 2.1.** The vector space  $\mathcal{A}$  carries a natural topology, and the set  $\text{Inv}(\mathcal{A})$  is open in  $\mathcal{A}$ .

**Remark 2.2.** The trace form  $\lambda$  of  $\mathcal{A}$  is an *associative* form, i.e.,  $\lambda(a, bc) = \lambda(ab, c)$  for all  $a, b, c \in \mathcal{A}$ .

Suppose that  $\lambda$  is *nondegenerate*. Then define the (not necessarily positive definite) line element  $ds^2$  by

$$ds^2 := \lambda(\dot{x}, P(x^{-1})\dot{x})dt^2,$$

**Remark 2.3.**  $ds^2$  is invariant under the maps  $x \rightarrow Wx$ ,  $W \in \Gamma(\mathcal{A})$ , and  $x \rightarrow x^{-1}$ .

In order to discuss the induced (pseudo-)Riemannian structure, let  $C$  be a connected component of  $\text{Inv}(\mathcal{A})$ .

**Proposition 2.1.** *There exists an  $f \in C$  such that  $f^2 = e$  and  $C = \text{Inv}_0(\mathcal{A})$ .*

**Remark 2.4.** Since  $\mathcal{A}_f$  is again a Jordan algebra (with unit element  $f^{-1}$ ), for the above mentioned discussion it suffices to consider  $\text{Inv}_0(\mathcal{A})$ .

**Theorem 2.1.** (Koecher) *If  $\mathcal{A}$  is a formally real Jordan algebra, then  $\text{Inv}_0(\mathcal{A})$  is a symmetric Riemannian space and:*

- (a) *at the point  $e$  the geodesic symmetry is the inversion  $x \rightarrow x^{-1}$ ;*
- (b) *the coefficients  $\Gamma_{jk}^i$  of the affine connection coincide with constants of the Jordan algebra  $\mathcal{A}$ .*

**Remark 2.5.** The space  $\text{Inv}_0(\mathcal{A})$  is a space with affine *constant* connection.

**Theorem 2.2.** (Sumitomo) *All 2-dimensional Riemannian spaces  $V_2$  with constant connection are locally Euclidean.*

**Theorem 2.3.** (Vagner) *If the metric of an  $n$ -dimensional Riemannian space  $V_n$  with constant connection is positive defined then it is of the form*

$$ds^2 = e^{a_k x^k} c_{ij} dx^i dx^j, \quad i, j, k = 1, \dots, n.$$

**Definition 2.1.** A Riemannian space  $V_n$  (with the metric positive definite or not) is called a *Vagner space* if its metric is like in Theorem 2.3.

**Theorem 2.4.** (Vranceanu) *A Riemannian space  $V_3$  with constant connection which is not locally Euclidean, is a Vagner space or has the metric of the form*

$$ds^2 = e^{2mz}(2dxdy + dy^2) \pm e^{4mz}dz^2.$$

**Theorem 2.5.** (Ruhnau) *The spaces of locally constant Levi-Civita connection are locally direct product of conformally flat Riemannian manifolds.*

To a space  $A_n$  with affine connection (particularly to a Riemannian space  $V_n$ ) one can associate a system of differential operators  $X_{kl}$  defined as follows

$$X_{kl} := \Gamma_{jkl}^i x^j \frac{\partial}{\partial x^i}.$$

**Remark 2.6.** When  $A_n$  (or  $V_n$ ) is with *constant* connection then the study of  $\{X_{kl}\}$  is equivalent to the study of the system of matrices  $M_{kl}$  defined by  $(M_{kl})_j^i := \Gamma_{jkl}^i$ .

**Definition 2.2.** A space  $V_n$  is called *irreducible (reducible)* in the point  $\mathcal{P}$  if the system  $\{X_{kl}\}$  which acts on the tangent space to  $A_n$  (or  $V_n$ ) at  $\mathcal{P}$  is irreducible (reducible). If the space  $V_n$  is irreducible in all its points then it is called *irreducible*.

**Remark 2.7.** The study of reducibility of a  $V_n$  with *constant connection* is equivalent with the study of reducibility of  $\{M_{kl}\}$  and if such a space is irreducible (reducible) in a point of it, then it is so in any other point of it.

**Theorem 2.6.** (Vranceanu) *If a Riemannian space is irreducible and has a positive definite metric, then it is determined, up to a constant factor, by  $\Gamma_{jk}^i$ .*

**Theorem 2.7.** (Teleman) *Under the same hypothesis like in Vranceanu's theorem, the metric is determined, up to a constant factor, by  $\Gamma_{jkl}^i$ .*

**Theorem 2.8.** (Iordanescu-Popovici-Turtoi) *The space  $A_{n+1}$  associated with Jordan algebra  $D_n(\eta_1, \dots, \eta_n)$  is a Vagner space with the metric*

$$ds^2 = e^{-2x^0} [(dx^0)^2 - \eta_1(dx^1)^2 - \dots - \eta_n(dx^n)^2],$$

which is determined, up to a constant factor, by  $\Gamma_{jk}^i$ .

**Notation.** By  $D_n(\eta_1, \dots, \eta_n)$  we have denoted the central simple algebra of type  $D$  with the basic  $\{e_0, e_1, \dots, e_n\}$  such that

$$e_i e_j = \eta_i \delta_{ij} e_0, \quad \eta_i = \pm 1, \quad i = 1, 2, \dots, n,$$

$$e_0 e_i = e_i e_0 = e_i, \quad i = 0, 1, 2, \dots, n.$$

**Open Problem.** Improve Vranceanu's and Teleman's Theorems using spaces given by the above theorem.

I would like to point out that the important contributions of the outstanding Romanian mathematicians Dan Barbilian and Gheorghe Tzitzeica are also involved in this topic.

The systematic study of projective planes over large classes of associative rings initiated by Dan Barbilian [2, 3] in 1940 has led – after more than thirty years – to the notions like Barbilian domains, Barbilian planes, Barbilian spaces, Barbilian geometry (see Section 3 of Chapter 3 from [36]). At present, in the Mathematical Subject Classification – which is everywhere accepted in the world - under the code 51C05 is mentioned his name. We must mention here the Romanian contribution to the study of Barbilian structures given by Francisc Radó (see Radó [76]).

The Tzitzeica surfaces (see Tzitzeica [80]), defined by himself in 1910, are connected with the inverse scattering method, which now is an important topic of theoretical physics research. The transformation that generates Tzitzeica surfaces and its slight generalizations (from the 1920s and the 1950s) are known in the modern literature on integrable equations as Darboux and Bäcklund transformations, which are now very important research tools (see Section 2 of Chapter 4 from [36]).

### 3. Applications of Jordan Structures to Dynamical Systems

Let us recall the construction given in 2000 by C.S. Liu (see Liu [62]).

If  $\mathbb{V}$  is a finite-dimensional real vector space endowed with an inner product denoted by the dot, then let us consider the real vector space  $\mathbb{M}$  as follows

$$\mathbb{M} := \{x_0 + gx_n \mid x_0 \in \mathbb{R}, x_n \in \mathbb{V}\}$$

with  $x + y := (x_0 + y_0) + g(x_n + y_n)$  for  $x, y \in \mathbb{M}$  and define also a product

$$xy = (x_0 + gx_n)(y_0 + gy_n) := (x_0 y_0 + x_n \cdot y_n) + g(x_0 y_n + y_0 x_n),$$

where  $g$  (called the “ $g$ -number” by C.S. Liu) satisfies

$$\begin{array}{c|cc} & 1 & g \\ \hline 1 & 1 & g \\ g & g & 1 \end{array}$$

It is easily to prove that, for  $x, y, z \in \mathbb{M}$ , and  $\lambda \in \mathbb{R}$ , we have

$$\begin{aligned} [x, y] &:= xy - yx = 0, & x(y + z) &= xy + xz, \\ (x + y)z &= xz + yz, & \lambda(x + y) &= \lambda x + \lambda y, \\ \lambda(xy) &= (\lambda x)y = x(\lambda y), & [x, y, z] &:= (xy)z - x(yz) = y[x_n, y_n, z_n] \neq 0, \\ [x^2, y, x] &= x^2(yx) - (x^2y)x = 0, & \text{where } x^2 &:= xx. \end{aligned}$$

**Remark 3.1.** Because we have  $(xy)z \neq x(yz)$ ,  $\mathbb{M}$  is a *non-associative* algebra.

**Remark 3.2.** Because we have  $[x, y] = 0$  and  $[x^2, y, x] = 0$ , the algebra  $\mathbb{M}$  is a *commutative Jordan algebra*.

**Remark 3.3.** Because we have  $x^2y \neq x(yx)$  and  $yx^2 \neq (yx)x$ , the Jordan algebra  $\mathbb{M}$  is *not alternative*.

**Remark 3.4.** The associator  $[x, y, z]$  is a *trilinear* product of  $\mathbb{M}$ .

Liu [62] has called  $\mathbb{M}$  the *g-based Jordan algebra*.

**Remark 3.5.** The *g*-based Jordan algebra  $\mathbb{M}$  is a particular *linear* Jordan algebra. It is the underling algebraic structure of a dynamical system defined on  $\mathbb{V}$  which possesses one or more constraints.

**Remark 3.6.** The associator of the *g*-based Jordan algebra  $\mathbb{M}$  can generate a vector field which includes one conservative force and one dissipative force.

**Remark 3.7.** Under certain conditions, the system may be refreshed as a generalized Hamiltonian system with singular non-canonical metric, or a metric system with degenerate Riemannian metric.

Some applications of this new formulation include the *perfect elastoplasticity* (see Hong and Liu [28, 29]), the *magnetic spin equation* (see Landau and Lifshitz [56]) the *suspension particle orientation equation* (see Liu [61]). They prove the *usefulness* of this new formulation of Liu.

**Open Problem.** As C.S. Liu suggested, there exists the possibility to describe the non-linear dissipative phenomena of physical systems by using the above mentioned *g*-based Jordan algebra  $\mathbb{M}$  (see Liu [62], p. 428).

One year later, in 2001, C.S. Liu has proceeded to examin above mentioned type of dynamical systems from the view point of Lie algebras and Lie groups. Then he has derived a new dynamical system based on the composition of the *g*-based Jordan algebra and Lie algebras (see Liu [63]).

**Remark 3.8.** Based on the symmetry study, C.S. Liu has developed a *numerical scheme* which preserves the group properties for every time increment.

**Important Remark.** Because the above mentioned scheme is easy to implement numerically and has high computational efficiency and accuracy, it is highly recommended for engineering applications.

In 2002, C.S. Liu has examined previous mentioned dynamical systems from the view point of Lie algebras and Lie groups (see Liu [64]).

**Open Problem.** Consider other *particular* Jordan algebras suitable to be the algebraic foundation for various dynamical systems.

On the other hand, it could be formulated also the following open problem.

**Open Problem.** Taking into account of the fact that the study of linear Jordan algebras can be included in the more general study of Jordan triple systems, it would be interesting to develop a *more general* algebraic background for (various) dynamical systems, and finally formulate an *unitary* mathematical theory for *all* dynamical systems.

Recently, in 2004, Liu [65] used the real  $g$ -based Jordan algebra (defined by himself in 2000) to the study of the Maxwell equations without appealing to the imaginary number  $i$ . In terms of the  $g$ -based Jordan algebra formulation, the usual Lorentz gauge condition is found to be a necessary and sufficient condition to render the second pair Maxwell equations, while the first pair of Maxwell equations is proved to be an intrinsic algebraic property.

The  $g$ -based Jordan and Lie algebras are a suitable system to implement to Maxwell equations into a more compact form.

Finally, C.S. Liu has studied in [64] the problem about a single formula of the Maxwell equations.

**Remark 3.9.** Forty years ago, in 1966, D. Hestenes has proved in his book [26] that – in terms of spacetime algebra (16 components) – the four Maxwell equations can be organized into a single one. Similarly, Liu [65] has achieved his goal in his algebraic formulation.

**Remark 3.10.** It is impressive that  $g$ -based Jordan (and Lie) algebras of Liu can be so useful to different topics.

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