This insertion is to replace pages 242 to 244, Vol.30, No.4, Gazette Aust Math Soc. which contains Prof Roger Eggleton's book review.
The Editors apologize for the errors contained in the bound version.

Abel's Proof

An Essay on the Sources and Meaning of Mathematical Unsolvability

Peter Pesic

MIT Press, 2003, pp.viii+213
ISBN 0 262 16216 4

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Peter Pesic is primarily a historian of science, having to his credit various scholarly writings on Francis Bacon and Gottfried Leibnitz. The star of his latest book is Abel, who gets title billing, but Galois is its costar, and numerous mathematical giants play significant parts in the development of its theme, among them Pythagoras, Euclid, al-Khwarizmi, Descartes, Newton, Gauss, Lagrange and Cauchy. But more than being about any person, this book is about a mathematical idea, the idea of unsolvability. Pesic mused (p.3): *[H]ow can a search for solutions yield the unsolvable? … I studied modern texts, but the key remained elusive. Absorbed in advanced studies, experts may cease to wonder about the elementary. They might not notice the kind of basic insight I was seeking. To find it, I needed to return to the sources.* So Pesic retraced the history of this idea, and *Abel's Proof* is his retelling of it: our understanding of the theme and appreciation of its significance grow as the historical developments unfold in context.

More specifically, the book is about the idea that polynomial equations in general cannot be solved exactly in radicals. The credit for proving this goes to Niels Henrik Abel, who showed in 1824 that fifth degree polynomials in general do not have solutions in radicals. Pesic asks (p.3): *What is it about the fifth degree that causes the problem? … Most of all, what is the significance of this breakdown, if one can use such a word?*

Pesic finds the origin of his theme in ancient Greece, with the invention of mathematical proof. Central to the Pythagorean world view was a notion of number as essentially integral, and length as essentially the ratio of whole multiples of units ("rational"), yet a proof was found that the side and diagonal of a square are incommensurable. Pesic says (p.10) this was *deeply disturbing, for it threatened the entire project of explaining nature in terms of number alone.* The philosophical crisis was only truly resolved many centuries later by the eventual acceptance of a wider, less intuitive notion of number in which the rational and the irrational stand side by side and have equal status.

A parallel thread in the evolving concept of number was the slow acceptance of negative numbers, called *absurd numbers* by some seventeenth century mathematicians. Even Descartes described them as *false or less than nothing*, and Laplace (1795) said the rule that the product of two negative numbers is a positive number *presents some difficulties.* The notion of a number line seems to have originated with Girard (1629), who accepted negative roots for equations, with the explanation that *the negative in geometry indicates a retrogression, where the positive is an advance.*

From origins in commerce and bookkeeping, the gradual emergence of algebra as a means of symbolic calculation was an essential mathematical development. Such symbol manipulation had to overcome the prejudice that it was mere sophistry and needed geometrical demonstration of its claims before one could confidently accept them. Pesic argues that Kepler and Galileo held opinions of this sort, and to some extent even Newton was inclined to such an attitude. But with algebra came the notion that mathematical problems could be formulated as equations to be solved. Thus the quest for exact solutions of polynomial equations gradually assumed the significance of a search for the solution of all mathematical problems, or as Viète (1595) put it, the motivation was *to leave no problem unsolved*.

Solution of particular quadratic equations by completing the square goes back to the Babylonians, though a general algebraic formulation, admitting not only negative roots but also complex roots, is a modern development. Pesic's account does not explicitly anchor the general solution of quadratic equations in history, but he does give a detailed historical treatment of the solution of cubic and quartic equations, wryly remarking (p.32) that contest over credit for the solution of the cubic *became the first example of a sordid modern genre: the scientific priority fight*. He gives helpful expositions of Cardano's method (pp.36–37) of solving cubic equations by completing the cube, and Ferrari's method (pp.38–39) of solving quartic equations by solving a cubic resolvent, then completing the square and solving two quadratics.

Failed attempts to adapt these methods to solve the quintic began to give hints that such methods could not possibly succeed. In particular, the cubic has a quadratic resolvent and the quartic has a cubic resolvent, but the resolvent of the quintic is not of smaller degree – in fact, its degree is 6. Lagrange concluded that new methods were needed to solve the quintic, but Gauss (1801) wrote that *there is little doubt that this problem does not so much defy modern methods of analysis as that it proposes the impossible*. At the same time Gauss gave the first proof that every polynomial equation has a root in the complex number plane, and hence a polynomial of degree n has n roots in the complex numbers. Thus the real question became: Is it true that the complex roots of a quintic cannot in general be expressed in terms of the coefficients by applying any finite sequence of additions, subtractions, multiplications, divisions and extractions of roots?

It seems that no contemporary mathematician produced such a clear and explicit formulation of the problem, but essentially that question motivated Ruffini (1799) to give a proof of the unsolvability of the quintic using Lagrange's work on permutations of the roots of a polynomial. Cauchy wrote that Ruffini's argument *proves completely the unsolvability of the general equation of degree greater than* 4, and went on (1813–15) to generalize some of his results on permutations. However, Lagrange contended that there were shortfalls in Ruffini's argument.

It fell to Abel (1824) to write the first proof which stood up to critical examination, and made clear the essential difference between equations of low degree and those of higher degree. Nonetheless, Pesic suggests (p.89) that the unsolvability theorem could justly be called the *Abel–Ruffini Theorem* because Ruffini's ideas contributed to Abel's argument via Cauchy's 1815 theorem (nicely expounded by Pesic in Appendix C). Pesic gives a commentary on Abel's proof, and includes an annotated translation of the original paper in Appendix A. The main ideas are (1) a characterization of the form of any solution in radicals; (2) proof that any such expression is a rational function of the roots (assumed without proof by Ruffini); (3) proof that if a rational function of five arguments takes fewer than five values when its arguments are permuted, then it takes at most two values; (4) showing that assuming the general quintic has a solution in radicals leads to the contradiction that two equal rational functions of the five roots take different numbers of values when the roots are permuted. By contradiction, it follows that the roots of the general quintic are not expressible in radicals. In modern terminology, there are algebraic numbers that are not derivable from the integers by any finite sequence of additions, subtractions, multiplications, divisions and extractions of roots. Pesic (p.146) favours Stewart's name *ultraradical numbers* for them.

Abel subsequently went on to investigate the conditions under which a polynomial equation can be solved in radicals, and found there are particular equations of all degrees, such as $x^n - 1 = 0$, that admit such solutions. Abel found that all the roots of such an equation are rational functions of each other, and if $f(a)$ and $g(a)$ are two such functions of a root a, then $f(g(a)) = g(f(a))$ so the two functions commute. This insight links solvability in radicals with commutativity of rational functions, and eventually led to the Jordan–Hölder Theorem (1889) on solvability in radicals.

Abel died of tuberculosis in 1829 at age 26, and the work of his last years was published posthumously. Unaware of that later work, and before his own death in 1832 at age 20, Galois famously generalized Abel's proof of unsolvability by great insights into the ideas about permutations of roots and their effects on the value of rational functions of those roots. Indeed, Galois was the first person to introduce the abstract notion of a group.

Pesic concludes his account after Abel and Galois by rather lightly tracing some of the development of ideas about commutativity and noncommutativity, and notes briefly (p.146) that following Abel, Jacobi, Hermite, Kronecker and Brioschi, in 1870 Jordan proved that elliptic modular functions suffice to solve all polynomial equations. The reader is left with little clarity on this sequel to the story, but with a very satisfactory understanding of Abel's proof.

The overall development in *Abel's Proof* is uneven and there are sporadic minor errors in the mathematical explanations, but these weaknesses are more than balanced by its excellent topically-organized notes. I give it three stars.