Path Integrals

Andreas Wipf

Theoretisch-Physikalisches-Institut

Friedrich-Schiller-Universität, Max Wien Platz 1

07743 Jena

5. Auflage WS 2008/09

1. Auflage, SS 1991 (ETH-Zürich)

I ask readers to report on errors in the manuscript and hope that the corrections will bring it closer to a level that students long for but authors find so elusive.

(email to: wipf@tpi.uni-jena.de) September 18, 2015

Contents

Chapter 1

Introduction

These lectures are intended as an introduction to path or functional integration techniques and their applications in physics. It is assumed that the participants have a good knowledge in quantum mechanics. No prior exposure to path integrals is assumed, however.

We are all familiar with the standard formulations of quantum mechanics, developed by HEISENBERG, SCHRÖDINGER and others in the 1920s. In 1933, DIRAC speculated that in quantum mechanic the classical action S might play a similarly important role as it does in classical mechanics. He arrived at the conclusion that the amplitude for the propagation from the initial position q' at time 0 to the final position q at time t,

$$K(t, q, q') = \langle q | e^{-iHt/\hbar} | q' \rangle, \tag{1.1}$$

is given by

$$K(t, q, q') \sim e^{iS[w_{\rm cl}]/\hbar}, \tag{1.2}$$

where $w_{\rm cl}$ is the classical trajectory from q' to q in time t. The exponent is dimensionless, since the reduced Planck-constant $\hbar$ has the dimension of an action. For a free particle with Hamiltonian and Lagrangian

$$H_0 = \frac{1}{2m} p^2 \quad \text{and} \quad L_0 = \frac{m}{2} \dot{q}^2 \tag{1.3}$$

the above formula is easily checked: free particles move on straight lines such that the trajectory $w(s)$ of a particle moving from q' to q and the corresponding action read

$$w(s) = \frac{1}{t} \{ sq + (t - s) q' \} \quad \text{and} \quad S = \int_0^t dt \, L_0(w, \dot{w}) = \frac{m}{2t} (q - q')^2. \tag{1.4}$$

Following Diracs suggestion this leads to the amplitude

$$K_0(t, q, q') \sim e^{im(q-q')^2/2\hbar t}. \tag{1.5}$$

The factor of proportionality can be inferred from the initial condition

$$e^{-iHt/\hbar} \stackrel{t\to 0}{\longrightarrow} \mathbb{1} \Longleftrightarrow \lim_{t\to 0} K(t,q,q') = \delta(q,q') \tag{1.6}$$

or alternatively from the convolution property

$$e^{-iHt/\hbar}e^{-iHs/\hbar} = e^{-iH(t+s)/\hbar}$$

which in position space takes the form

$$\int du\, K(t,q,u)K(s,u,q') = K(t+s,q,q'). \tag{1.7}$$

Both ways one arrives at the propagator for a free particle,

$$K_0(t,q,q') = \Big(\frac{m}{2\pi i\hbar t}\Big)^{1/2}\, e^{iS[w_{\rm cl}]/\hbar}. \tag{1.8}$$

As we shall see later, similar results hold true for motions in harmonic potentials, for which $\langle V'(\hat q)\rangle = V'(\langle \hat q\rangle)$, such that $\langle \hat q\rangle$ satisfies the classical equation of motion.

However, for nonlinear systems the formula (1.8) is modified. In 1948 FEYNMAN succeeded in extending Diracs result to interacting systems. He found an alternative formulation of quantum mechanics, based on the fact that the propagator can be written as a sum over *all possible paths* (and not just the classical paths) from the initial to the final point. One may say that in quantum mechanics a particle may move along any path $w(t)$ connecting the initial with the final point in time t,

$$w(0) = q' \quad \text{and} \quad w(t) = q. \tag{1.9}$$

The amplitude for an individual path is $\sim \exp\left(iS[\text{path}]/\hbar\right)$ and the amplitudes for all paths are added according to the usual rule for combining probability amplitudes,

$$K(t,q,q') \sim \sum_{\text{paths } q'\to q} e^{iS[\text{path}]/\hbar}. \tag{1.10}$$

Surprisingly enough, the same calculus (in the sense of a analytical continuation) was already known to mathematicians due to the work of WIENER in the study of stochastic processes. This *calculus* in functional space attracted the attention of other mathematicians, including KAC, and was subsequently further developed. The standard reference concerning these achievements is the review of GELFAND and YAGLOM [5], where the early work was first critically discussed.

The path integral method had its great, early successes in the 1950s and its implications have been beautifully expounded in Feynmans original review paper [3] and in his book with HIBBS [4]. This book contains many applications and still serves as a standard literature on path integrals.

Path integration provides a *unified view* of quantum mechanics, field theory and statistical physics and is nowadays a irreplaceable tool in theoretical physics. It is an alternative to the Hamiltonian method for quantizing classical systems and solving problems in quantum mechanics and quantum field theories.

These lectures should introduce you both into the formalism and the techniques of path integration. We shall discuss applications that will convince you that path integrals are worth studying not only for reasons of beauty but also for practical purposes.

Path integrals in quantum mechanics and quantum field theory are ideally suited to deal with problems like

- Implementing symmetries of a theory
- Incorporating constraints
- Studying non-perturbative effects
- Deriving the semiclassical approximation
- Describing finite-temperature field theories
- Connecting quantum field theories to statistical systems
- Renormalization and renormalization group transformations
- Numerical simulations of field theories.

In the first part of these lecture we shall reformulate ordinary quantum mechanics in Feynmans path integral language. We shall see how to manipulate path integrals and we shall apply the results to simple physical systems: the *harmonic oscillator* with constant and time dependent frequency and the driven oscillator. Then we consider the path integral for imaginary time and give a precise meaning to the sum over all paths. *Functional determinants* show up in many path integral manipulations and we devote a whole section to these objects. It follows a chapter on the path integral approach to quantum systems in thermal equilibrium. We derive the semiclassical and high-temperature expansions to the partition functions and conclude the part on quantum mechanics with Monte Carlo simulations of discretized Quantum Mechanics.

In the second part these lectures a simple field-theoretical model, namely the Schwinger model or QED in 2 dimensions, is introduced and solved. This model is interesting for various reasons. Due to quantum correction the 'photon' acquires a mass and the classical chiral symmetry is broken like it is in QCD. These model allows us to introduce many relevant field theoretical concepts like regularization, Berezin-integrals, gauge fixing and perturbation theory. Then we deal with anomalies and effective actions. We shall see how to employ path integral techniques to compute anomalies in gauge theories. We 'integrate' certain anomalies and derive the Casimir effect in external fields. Finally we shall compute the particle production in external electromagnetic and gravitational fields.

In the last part of these lectures we study the lattice version of field theories. In particular we introduce and discuss the symmetry breaking by means of effective potentials. Then the numerical simulations of scalar theories on a finite lattice is discussed. Finally I shall explain how to formulate gauge field theories with fermions on a space-time lattice and the some problems of these lattice gauge theories.

There are many good books and review articles on path integrals. I have listed some references which I suggest for further readings. In particular the references [1]-[9] contain introductory material. These references are only a very small and subjective selection from the extensive literature on functional integrals. In the bibliography at the end of these lectures you find further references on particular topics of path integrals.

The path integral formulation of quantum mechanics is a description of quantum theory that generalizes the action principle of classical mechanics. It replaces the classical notion of a single, unique classical trajectory for a system with a sum, or functional integral, over an infinity of quantum-mechanically possible trajectories to compute a quantum amplitude. A sizable fraction of the theoretical developments in physics of the last sixty years would not be understandable without the use of path or, more generally, field integrals. In this article we will focus on the use of path integrals and field integrals in different branches of theoretical physics. A rigorous study of the mathematical properties of path and field integrals is an open subtopic of functional analysis and will not be dealt with here.