Surveys of Modern Mathematics
Volume I

Analytic Methods in Algebraic Geometry

by Jean-Pierre Demailly

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Analytic Methods in Algebraic Geometry

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Surveys of Modern Mathematics

Preface

The main purpose of this book is to describe analytic techniques which are useful to study questions such as linear series, multiplier ideals and vanishing theorems for algebraic vector bundles. One century after the ground-breaking work of Riemann on geometric aspects of function theory, the general progress achieved in differential geometry and global analysis on manifolds resulted into major advances in the theory of algebraic and analytic varieties of arbitrary dimension. One central unifying concept is positivity, which can be viewed either in algebraic terms (positivity of divisors and algebraic cycles), or in more analytic terms (plurisubharmonicity, Hermitian connections with positive curvature). In this direction, one of the most basic results is Kodaira's vanishing theorem for positive vector bundles (1953—1954), which is a deep consequence of the Bochner technique and the theory of harmonic forms initiated by Hodge during the 1940's. This method quickly led Kodaira to the well-known embedding theorem for projective varieties, a far reaching extension of Riemann's characterization of abelian varieties. Further refinements of the Bochner technique led ten years later to the theory of L^2 estimates for the Cauchy-Riemann operator, in the hands of Kohn, Andreotti-Vesentini and Hörmander among others. Not only can vanishing theorems be proved or reproved in that manner, but perhaps more importantly, extremely precise information of a quantitative nature can be obtained about solutions of $\overline{\partial}$-equations, their zeroes, poles and growth at infinity.

We try to present here a condensed exposition of these techniques, assuming that the reader is already somewhat acquainted with the basic concepts pertaining to sheaf theory, cohomology and complex differential geometry. In the final chapter, we address very recent questions and open problems, e.g. results related to the finiteness of the canonical ring and the abundance conjecture, as well as results describing the geometric structure of Kähler varieties and their positive cones.

This book is an expansion of lectures given by the author at the Park City Mathematics Institute in 2008 and was published partly in Analytic and Algebraic Geometry, edited by Jeff McNeal and Mircea Mustaţă, It is a volume in the Park City Mathematics Series, a co-publication of the Park City Mathematics Institute and the American Mathematical Society.

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Contents

Introduction

This introduction will serve as a general guide for reading the various parts of this text. The first three chapters briefly introduce basic materials concerning complex differential geometry, Dolbeault cohomology, plurisubharmonic functions, positive currents and holomorphic vector bundles. They are mainly intended to fix notation. Although the most important concepts are redefined, readers will probably need to already possess some related background in complex analysis and complex differential geometry — whereas the expert readers should be able to quickly proceed further.

The heart of the subject starts with the Bochner technique in Chapter 4, leading to fundamental L^2 existence theorems for solutions of $\overline{\partial}$-equations in Chapter 5. What makes the theory extremely flexible is the possibility to formulate existence theorems with a wide assortment of different L^2 norms, namely norms of the form $\int_X |f|^2 e^{-2\varphi}$ where φ is a plurisubharmonic or strictly plurisubharmonic function on the given manifold or variety X. Here, the weight φ need not be smooth, and on the contrary, it is extremely important to allow weights which have logarithmic poles of the form $\varphi(z) = c \log \sum |g_j|^2$, where $c > 0$ and (g_j) is a collection of holomorphic functions possessing a common zero set $Z \subset X$. Following Nadel [Nad89], one defines the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ to be the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally summable. Then $\mathcal{I}(\varphi)$ is a coherent algebraic sheaf over X and $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ for all $q \geqslant 1$ if the curvature of L is positive as a current. This important result can be seen as a generalization of the Kawamata-Viehweg vanishing theorem [Kaw82, Vie82], which is one of the cornerstones of higher dimensional algebraic geometry, especially in relation with Mori's minimal model program.

In the dictionary between analytic geometry and algebraic geometry, the ideal $\mathcal{I}(\varphi)$ plays a very important role, since it directly converts an analytic object into an algebraic one, and, simultaneously, takes care of the singularities in a very efficient way. Another analytic tool used to deal with singularities is the theory of positive currents introduced by Lelong [Lel57]. Currents can be seen as generalizations of algebraic cycles, and many classical results of intersection theory still apply to currents. The concept of Lelong number of a current is the analytic analogue of the concept of multiplicity of a germ of algebraic variety. Intersections of cycles correspond to wedge products of currents (whenever these products are defined).

Besides the Kodaira-Nakano vanishing theorem, one of the most basic "effective result" expected to hold in algebraic geometry is expressed in the following conjecture of Fujita [Fuj87]: if L is an ample (i.e. positive) line bundle on a projective n-dimensional algebraic variety X, then $K_X + (n+1)L$ is generated by sections and $K_X + (n+2)L$ is very ample. In the last two decades, a lot of efforts have been brought for the solution of this conjecture — but reaching the expected optimal bounds will probably require new ideas. The first major results are the proof of the Fujita conjecture in the case of surfaces by Reider [Rei88] (the case of curves is easy and has been known since a very long time), and the numerical criterion for the very ampleness of $2K_X + L$ given in [Dem93b], obtained by means of analytic techniques and Monge-Ampère equations with isolated singularities. Alternative algebraic techniques were developed slightly later by Kollár [Kol92], Ein-Lazarsfeld [EL93], Fujita [Fuj93], Siu [Siu95, 96], Kawamata [Kaw97] and Helmke [Hel97]. We will explain here Siu's method because it is technically the simplest method; one of the results obtained by this method is the following effective result: $2K_X + mL$ is very ample for $m \geqslant 2 + \binom{3n+1}{n}$. The basic idea is to apply the Kawamata-Viehweg vanishing theorem, and to combine this with the Riemann-Roch formula in order to produce sections through a clever induction procedure on the dimension of the base loci of the linear systems involved.

Although Siu's result is certainly not optimal, it is sufficient to obtain a nice constructive proof of *Matsusaka's big theorem* [Siu93, Dem96]. The result states that there is an effective value m_0 depending only on the intersection numbers L^n and $L^{n-1} \cdot K_X$, such that mL is very ample for $m \geqslant m_0$. The basic idea is to combine results on the very ampleness of $2K_X + mL$ together with the theory of holomorphic Morse inequalities [Dem85b]. The Morse inequalities are used to construct sections of $m'L - K_X$ for m' large. Again this step can be made algebraic (following suggestions by F. Catanese and R. Lazarsfeld), but the analytic formulation apparently has a wider range of applicability.

In the subsequent chapters, we pursue the study of L^2 estimates, in relation with the Nullstellenstatz and with the extension problem. Skoda [Sko72b, 78] showed that the division problem $f = \sum g_j h_j$ can be solved holomorphically with very precise L^2 estimates, provided that the L^2 norm of $|f|\,|g|^{-p}$ is finite for some sufficiently large exponent p ($p > n = \dim X$ is enough). Skoda's estimates have a nice interpretation in terms of local algebra, and they lead to precise qualitative and quantitative estimates in connection with the Bézout problem. Another very important result is the L^2 extension theorem by Ohsawa-Takegoshi [OT87, Ohs88], which has also been generalized later by Manivel [Man93]. The main statement is that every L^2 section f of a suitably positive line bundle defined on a subavariety $Y \subset X$ can be extended to a L^2 section $\tilde{f}$ defined over the whole of X. The positivity condition can be understood in terms of the canonical sheaf and normal bundle to the subvariety. The extension theorem turns out to have an incredible amount of important consequences: among them, let us mention for instance Siu's theorem [Siu74] on the analyticity of Lelong numbers, the basic approximation theorem of closed positive $(1,1)$-currents by divisors, the subadditivity property $\mathcal{I}(\varphi + \psi) \subset \mathcal{I}(\varphi)\mathcal{I}(\psi)$ of multiplier ideals [DEL00], the restriction formula $\mathcal{I}(\varphi_{|Y}) \subset \mathcal{I}(\varphi)_{|Y}$, ... A suitable combination of these results yields another important result of Fujita [Fuj94] on approximate Zariski decomposition, as we show in Chapter 14.

In Chapter 15, we show how subadditivity can be used to derive an "equisingular" approximation theorem for (almost) plurisubharmonic functions: any such function can be approximated by a sequence of (almost) plurisubharmonic functions which are smooth outside an analytic set, and which define the same multiplier ideal sheaves. From this, we derive a generalized version of the *hard Lefschetz theorem* for cohomology with values in a pseudo-effective line bundle; namely, the Lefschetz map is surjective when the cohomology groups are twisted by the relevant multiplier ideal sheaves.

Chapter 16 explains the proof of Siu's theorem on the invariance of plurigenera, according to a beautiful approach developped by Mihai Păun [Pău07]. The proofs consists of an iterative process based on the Ohsawa-Takegoshi theorem, and a very clever limiting argument for currents.

Chapters 17 and 18 are devoted to the study of positive cones in Kähler or projective geometry. Recent "algebro-analytic" characterizations of the Kähler cone [DP04] and the pseudo-effective cone of divisors [BDPP04] are explained in detail. This leads to a discussion of the important concepts of volume and mobile intersections, following S. Boucksom's PhD work [Bou02]. As a consequence, we show that a projective algebraic manifold has a pseudo-effective canonical line bundle if and only if it is not uniruled.

Chapter 19 presents further important ideas of H. Tsuji, later refined by Berndtsson and Păun, concerning the so-called "super-canonical metrics", and their interpretation in terms of the invariance of plurigenera and of the abundance conjecture. In the concluding Chapter 20, we state Păun's version of the Shokurov-Hacon-McKernan-Siu non vanishing theorem and give an account of the very recent approach of the proof of the finiteness of the canonical ring by Birkar-Păun [BiP09], based on the ideas of Hacon-McKernan and Siu.

Analytic geometry arose with the importing of algebraic notions and notations into geome-try. Descartes, at least, justied the algebra geometrically. Now it is possible to go the other way, using algebra to justify geometry.Â Thus one can see clearly that Theorem is true, without needing to resort to any of the analytic methods of the rst four proofs. . Failures of rigor. The root meaning of the word â€œrigorâ€ is stiness. I will present both algebraic and analytic constructions, inspired by those for multiplier ideals. Using Kodaira-Saito vanishing, I will prove a Nadel-type vanishing theorem for multiplier subsheaves, generalizing a number of vanishing theorems in algebraic geometry. If time permits, I will present an application to a Fujita-type freeness result for the lowest term in the Hodge filtration. 2.30pm - 3.30pm. GÃ¡bor SzÃ©kelyhidi (Notre Dame) The KÃ¤hler-Ricci flow and optimal degenerations - Abstract. Chen-Sun-Wang showed that the KÃ¤hler-Ricci flow on a Fano manifold gives rise to a certain algebrai...