

# MATHEMATICAL MAYHEM

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## Shreds and Slices

### The K Method

In this brief article, we will present a useful geometrical tool, which we coin “The K Method”. As is convention, let  $[P]$  denote the area of polygon  $P$ . We will further stipulate that if  $P$  is labelled counter-clockwise, then  $[P]$  is positive, and negative otherwise. This sign convention will matter.

Let  $ABC$  be a triangle, labelled counter-clockwise, and let  $P$  be a point in the plane. Let  $K = [ABC]$ ,  $K_A = [PBC]$ ,  $K_B = [PCA]$ , and  $K_C = [PAB]$  (see Figure 1). Because of the signed areas, we have that for all points  $P$ , inside or outside the triangle  $ABC$ ,

$$K = K_A + K_B + K_C.$$

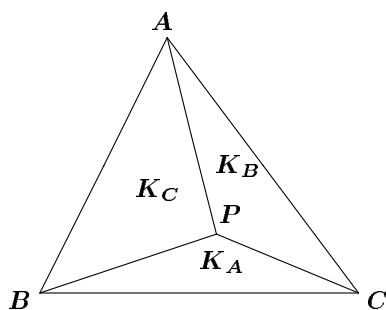


Figure 1

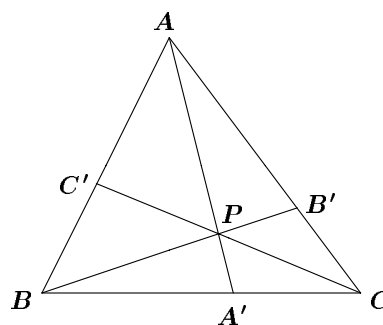


Figure 2

Now, extend  $AP$  to  $A'$  on  $BC$ , and define  $B'$  and  $C'$  similarly (see Figure 2). Recall that triangles with the same height have areas in proportion to their bases. Then we have

$$\frac{AP}{PA'} = \frac{[PAB]}{[PBA']} = \frac{[PCA]}{[PA'C]} = \frac{[PAB] + [PCA]}{[PBA'] + [PA'C]} = \frac{K_B + K_C}{K_A}, \text{ and}$$

$$\frac{BA'}{A'C} = \frac{[ABA']}{[AA'C]} = \frac{[PBA']}{[PA'C]} = \frac{[ABA'] - [PBA']}{[AA'C] - [PA'C]} = \frac{K_C}{K_B}.$$

We can similarly derive that

$$\frac{BP}{PB'} = \frac{K_A + K_C}{K_B}, \quad \frac{CP}{PC'} = \frac{K_A + K_B}{K_C}, \quad \frac{CB'}{B'A} = \frac{K_A}{K_C}, \text{ and } \frac{AC'}{C'B} = \frac{K_B}{K_A}.$$

Again, these hold regardless of whether  $P$  is inside or outside triangle  $ABC$ , but only with directed line segments (that is, if  $PQ$  and  $PR$  are in different directions, then their ratio will be negative).

These expressions can be very useful in problems which involve these ratios. In fact, one half of Ceva's theorem is now trivial (and with some consideration, so is the other half).

**Problem 1.** Consider triangle  $P_1P_2P_3$  and a point  $P$  within the triangle. Lines  $P_1P$ ,  $P_2P$ ,  $P_3P$  intersect the opposite sides in points  $Q_1$ ,  $Q_2$ ,  $Q_3$  respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \quad \frac{P_2P}{PQ_2}, \quad \frac{P_3P}{PQ_3},$$

at least one is less than or equal to 2 and at least one is greater than or equal to 2. (1961 IMO, Problem #4)

**Solution.** Let us use the same understood notation as above. Without loss of generality, we can assume that  $K_A \leq K_B \leq K_C$  (re-label the triangle if necessary). Then

$$\frac{P_3P}{PQ_3} = \frac{K_A + K_B}{K_C} \leq \frac{K_C + K_C}{K_C} = 2, \text{ and}$$

$$\frac{P_1P}{PQ_1} = \frac{K_B + K_C}{K_A} \geq \frac{K_A + K_A}{K_A} = 2.$$

**Problem 2.** In triangle  $ABC$ ,  $A'$ ,  $B'$ , and  $C'$  are on sides  $BC$ ,  $AC$ , and  $AB$ , respectively. Given that  $AA'$ ,  $BB'$ , and  $CC'$  are concurrent at the point  $O$ , and that  $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$ , find the value of

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}. \quad (1992 \text{ AIME})$$

**Solution.** The given implies

$$\frac{K_A + K_B}{K_C} + \frac{K_A + K_C}{K_B} + \frac{K_B + K_C}{K_A} = 92,$$

and further that

$$K_A^2 K_B + K_A K_B^2 + K_A^2 K_C + K_A K_C^2 + K_B^2 K_C + K_B K_C^2 = 92 K_A K_B K_C.$$

Then

$$\begin{aligned} \frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} &= \left( \frac{K_B + K_C}{K_A} \right) \left( \frac{K_A + K_C}{K_B} \right) \left( \frac{K_A + K_B}{K_C} \right) \\ &= \frac{K_A^2 K_B + K_A K_B^2 + K_A^2 K_C + K_A K_C^2 + K_B^2 K_C + K_B K_C^2 + 2K_A K_B K_C}{K_A K_B K_C} \\ &= \frac{92K_A K_B K_C + 2K_A K_B K_C}{K_A K_B K_C} = 94. \end{aligned}$$

## Problems

1. Prove Ceva's Theorem, which states that  $AA'$ ,  $BB'$ , and  $CC'$  (as in Figure 2) are collinear if and only if

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

2. Let  $P$  be a point inside the triangle  $ABC$ . Let  $AP$  meet  $BC$  at  $D$ ,  $BP$  meet  $CA$  at  $E$ , and  $CP$  meet  $AB$  at  $F$ . Prove that

$$\frac{PA}{PD} \cdot \frac{PB}{PE} + \frac{PB}{PE} \cdot \frac{PC}{PF} + \frac{PC}{PF} \cdot \frac{PA}{PD} \geq 12.$$

(*The Red Book of Mathematical Problems*, Williams and Hardy)



## Powers of Two

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We will solve several problems involving the number 2 and, in the process, survey various interesting mathematical results.

**Problem 1.** Prove that for all positive integers  $n$ ,  $p_n(x) = x^n - 2$  is irreducible in  $\mathbb{Q}[x]$ .

*Remark.*  $\mathbb{Q}[x]$  is the set (or more precisely, ring) of polynomials in  $x$  with rational coefficients. Hence, a polynomial is irreducible in  $\mathbb{Q}[x]$  if and only if it cannot be factored non-trivially as a product of polynomials, also with rational coefficients.

**Solution.** This follows directly from Eisenstein's Criterion, but we will take a more basic approach. By DeMoivre's Theorem, the  $n$  roots of  $p_n(x)$  (in  $\mathbb{C}$ ) are  $2^{\frac{1}{n}} \text{cis}(\frac{2k\pi}{n})$ ,  $k = 0, 1, 2, \dots, n-1$ , so that

$$p_n(x) = \left(x - 2^{\frac{1}{n}} \text{cis} 0\right) \left(x - 2^{\frac{1}{n}} \text{cis} \left(\frac{2\pi}{n}\right)\right) \cdots \left(x - 2^{\frac{1}{n}} \text{cis} \left(\frac{2(n-1)\pi}{n}\right)\right).$$

Suppose  $p_n(x) = f(x)g(x)$  for some  $f(x), g(x) \in \mathbb{Q}[x]$ , with  $\deg f(x) = m$  and  $n > m > 0$ . (Note: we only need to consider  $n \geq 2$ .) Let the roots of  $f(x)$ , respectively  $g(x)$ , be  $\omega_1, \omega_2, \dots, \omega_m$ , respectively  $\omega_{m+1}, \omega_{m+2}, \dots, \omega_n$ , so  $\omega_1, \omega_2, \dots, \omega_n$  is a permutation of the roots of  $p_n(x)$ . Let  $\omega_i = 2^{1/n} \text{cis} \theta_i$ , and let  $\theta = \sum_{k=1}^m \theta_k$ . Note  $(-1)^m \prod_{k=1}^m \omega_k$  is the constant term of  $f(x)$ , so

$$(-1)^m \prod_{k=1}^m \omega_k = (-1)^m 2^{\frac{m}{n}} \prod_{k=1}^m \text{cis} \theta_k = (-1)^m 2^{\frac{m}{n}} \text{cis} \theta \in \mathbb{Q}.$$

But  $\text{cis} \theta = \cos \theta + i \sin \theta \in \mathbb{Q}$  implies that  $\sin \theta = 0$  and further, that  $\cos \theta = \pm 1$ . Therefore, the constant term of  $f(x)$  is  $\pm 2^{\frac{m}{n}}$ . It is easy to show that  $2^{\frac{m}{n}} \notin \mathbb{Q}$ , a contradiction. Hence,  $x^n - 2$  is irreducible in  $\mathbb{Q}[x]$ .

**Problem 2.** If the sum of the proper divisors of  $n$  (that is, the divisors of  $n$  that are less than  $n$ ) is  $n$ , then  $n$  is called a *perfect number*. Equivalently, the sum of the divisors of  $n$ ,  $\sigma(n)$ , is  $2n$ . For example,  $2 \cdot 6 = \sigma(6) = 1 + 2 + 3 + 6$ . Classify all even perfect numbers.

**Solution.** By the Fundamental Theorem of Arithmetic, we can express  $n$  uniquely, up to permutations, in the form  $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ , where the  $p_i$  are

distinct primes and the  $a_i$  are positive integers. Then, all divisors of  $n$  are of the form  $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ , where  $0 \leq \alpha_i \leq a_i$ . Hence,

$$\begin{aligned}\sigma(n) &= \prod_{j=1}^k (1 + p_j + p_j^2 + \cdots + p_j^{a_j}) \\ &= \left( \frac{p_1^{a_1+1} - 1}{p_1 - 1} \right) \left( \frac{p_2^{a_2+1} - 1}{p_2 - 1} \right) \cdots \left( \frac{p_k^{a_k+1} - 1}{p_k - 1} \right).\end{aligned}$$

It follows that if  $\gcd(m, n) = 1$ , then  $\sigma(mn) = \sigma(m)\sigma(n)$ .

We claim that if  $2^n - 1$  is prime, then  $2^{n-1}(2^n - 1)$  is a perfect number. In such a case, we have

$$\begin{aligned}\sigma(2^{n-1}(2^n - 1)) &= \frac{2^n - 1}{2 - 1} \cdot \frac{(2^n - 1)^2 - 1}{(2^n - 1) - 1} \\ &= (2^n - 1)(2^n - 1 + 1) = 2 \cdot 2^{n-1}(2^n - 1).\end{aligned}$$

Now consider an arbitrary even perfect number  $n$ . Then  $n = 2^k m$ , where  $k \geq 1$  and  $\gcd(2^k, m) = 1$ . Also,  $2^{k+1}m = \sigma(2^k m) = \sigma(2^k)\sigma(m) = (2^{k+1} - 1)\sigma(m)$ . Now,  $2^{k+1} - 1$  is odd, so  $(2^{k+1} - 1) \mid m$ . Let  $m = d(2^{k+1} - 1)$ , so  $d$  is a divisor of  $m$ . By the above,

$$\sigma(m) = \frac{2^{k+1}m}{2^{k+1} - 1} = m + \frac{m}{2^{k+1} - 1} = m + d,$$

but  $\sigma(m)$  is defined as the sum of *all* of the divisors of  $m$ . Therefore,  $m$  only has the two divisors  $m$  and  $d$ , so  $m$  is prime,  $d$  is 1, and so  $2^{k+1} - 1$  is prime.

We have shown that all even perfect numbers are of the form  $2^{n-1}(2^n - 1)$ , where  $2^n - 1$  is prime. Primes of this form are called *Mersenne primes*.

**Problem 3.** Perform the following transformation  $T$  on the vector  $(x_1, x_2, x_3, \dots, x_{2^n})$ , where the  $x_i$  are non-negative integers:

$$T(x_1, x_2, x_3, \dots, x_{2^n}) = (|x_1 - x_2|, |x_2 - x_3|, \dots, |x_{2^n} - x_1|).$$

Prove that a finite number of applications of  $T$  will transform any such vector to the zero vector.

**Solution 1.** Notice that mod 2 has the following special property:  $1 \equiv -1 \pmod{2} \Rightarrow a \equiv -a \pmod{2}$  for all integers  $a$ . This implies  $|a| \equiv a \pmod{2}$  for all integers  $a$ , and that subtraction is the same as addition in mod 2. Therefore, the values  $|x_1 - x_2|, |x_2 - x_3|, \dots, |x_{2^n} - x_1| \pmod{2}$ , are congruent to  $x_1 + x_2, x_2 + x_3, \dots, x_{2^n} + x_1 \pmod{2}$ , which will be easier to work with.

Define  $S(x_1, x_2, \dots, x_{2^n}) = (x_1 + x_2, x_2 + x_3, \dots, x_{2^n} + x_1)$ . By the previous remarks,  $S$  and  $T$  give the same results mod 2. Notice that

$$\begin{aligned} S^2(x_1, x_2, x_3, \dots, x_{2^n}) \\ = (x_1 + 2x_2 + x_3, x_2 + 2x_3 + x_4, \dots, x_{2^n} + 2x_1 + x_2), \end{aligned}$$

and

$$\begin{aligned} S^3(x_1, x_2, \dots, x_{2^n}) \\ = (x_1 + 3x_2 + 3x_3 + x_4, x_2 + 3x_3 + 3x_4 + x_5, \\ \dots, x_{2^n} + 3x_1 + 3x_2 + x_3). \end{aligned}$$

The coefficients seem to be entries in Pascal's triangle. Indeed, we can prove by induction that the  $i^{\text{th}}$  entry of  $S^k(x_1, x_2, x_3, \dots, x_{2^n})$  is  $\sum_{j=0}^k \binom{k}{j} x_{j+i}$ , where  $x_r = x_s$  if  $r \equiv s \pmod{2^n}$ .

What does  $S^{2^n}(x_1, x_2, x_3, \dots, x_{2^n})$  look like? First, we claim that if  $1 \leq j \leq 2^n - 1$ , then  $\binom{2^n}{j}$  is even. We prove the more general result which states that if  $p$  is prime and  $1 \leq j \leq p^k - 1$ , then  $p$  divides  $\binom{p^k}{j}$ .

Step 1: For  $p$  prime, we determine the highest power of  $p$  dividing  $m!$ .

There are  $\lfloor \frac{m}{p} \rfloor$  natural numbers less than or equal to  $m$  that are divisible by  $p$ ,  $\lfloor \frac{m}{p^2} \rfloor$  natural numbers less than or equal to  $m$  that are divisible by  $p^2$ , and so on. For each number divisible by at most  $p^a$ , we have counted it exactly  $a$  times in the sums.

Therefore, the highest power of  $p$  dividing  $m!$  is

$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \dots$$

Step 2: We calculate the highest power of  $p$  dividing  $\binom{p^k}{j}$ , where  $1 \leq j \leq p^k - 1$  and  $p$  is prime, which is simply the highest power of  $p$  dividing the numerator subtracted by the highest power of  $p$  dividing the denominator.

Applying the result from the previous step, the highest power of  $p$  dividing  $\binom{p^k}{j} = \frac{p^k!}{j!(p^k-j)!}$  is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{p^k}{p^i} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{j}{p^i} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{p^k - j}{p^i} \right\rfloor = \sum_{i=1}^k \left( \left\lfloor \frac{p^k}{p^i} \right\rfloor - \left\lfloor \frac{j}{p^i} \right\rfloor - \left\lfloor \frac{p^k - j}{p^i} \right\rfloor \right).$$

Each term in the sum is non-negative, since  $\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$  for all reals  $x, y$ , and when  $i = k$ ,

$$\left\lfloor \frac{p^k}{p^k} \right\rfloor - \left\lfloor \frac{j}{p^k} \right\rfloor - \left\lfloor \frac{p^k - j}{p^k} \right\rfloor = 1,$$

since  $1 \leq j \leq p^k - 1$ . Hence, there is at least one factor of  $p$  in  $\binom{p^k}{j}$ .

Now we relate this result to the problem. The parity of the  $i^{\text{th}}$  entry of  $T^{2^n}(x_1, x_2, x_3, \dots, x_{2^n})$  is the same as the parity of the  $i^{\text{th}}$  entry of  $S^{2^n}(x_1, x_2, x_3, \dots, x_{2^n})$ , which is

$$\sum_{j=0}^{2^n} \binom{2^n}{j} x_{j+i} \equiv \binom{2^n}{0} x_i + \binom{2^n}{2^n} x_{i+2^n} \equiv 2x_i \equiv 0 \pmod{2}.$$

Therefore, the  $i^{\text{th}}$  entry of  $T^{2^n}(x_1, x_2, x_3, \dots, x_{2^n})$  is even. We can then pull a factor of 2 out of each of the resulting elements and apply  $T$   $2^n$  times to that vector (since  $T$  is linear), and again, obtain a vector with all entries divisible by 2, and so on. Hence, after  $t \cdot 2^n$  applications of  $T$ , each entry in the vector must be divisible by  $2^t$ .

If  $y_k$  is used to denote the largest of the  $2^n$  entries of

$$T^k(x_1, x_2, x_3, \dots, x_{2^n}),$$

then  $\{y_k\}$  is a non-increasing sequence. Furthermore, there exists  $t$  such that  $y_1 < 2^t$ . Then after  $t \cdot 2^n$  applications of  $T$ , the resulting vector has entries all divisible by  $2^t$ . Since the entries are non-negative and less than  $2^t$ , they must all be zero. Therefore, we have transformed the original vector to the zero vector with a finite number of applications of  $T$ .

**Solution 2.** We shall prove by induction that for all natural  $n$ , there exists  $k$  such that every entry of  $T^k(x_1, x_2, x_3, \dots, x_{2^n})$  and, equivalently, every entry of  $S^k(x_1, x_2, x_3, \dots, x_{2^n})$  is even for all non-negative integers  $x_1, x_2, \dots, x_{2^n}$ . When  $n = 0$ ,  $T(x_1) = x_1 - x_1 = 0$  and our hypothesis holds. Assume the result holds for some  $n = r$ , with  $k_r$  iterations of  $T$  always producing a result whose entries are all divisible by 2. Let  $n = r + 1$ . Then

$$\begin{aligned} & S(x_1, x_2, x_3, \dots, x_{2^{r+1}}) \\ &= (x_1 + x_2, x_2 + x_3, x_3 + x_4, \dots, x_{2^{r+1}} + x_1), \\ & S^2(x_1, x_2, x_3, \dots, x_{2^{r+1}}) \\ &= (x_1 + 2x_2 + x_3, x_2 + 2x_3 + x_4, x_3 + 2x_4 + x_5, \dots) \\ &\equiv (x_1 + x_3, x_2 + x_4, x_3 + x_5, \dots, x_{2^{r+1}} + x_2) \pmod{2}. \end{aligned}$$

If we take every other element of the last vector, starting with the first, we obtain  $(x_1 + x_3, x_3 + x_5, \dots, x_{2^{r+1}-1} + x_1)$ , which is also what we obtain when we apply  $T$  to  $(x_1, x_3, x_5, \dots, x_{2^{r+1}-1})$ , and similarly with the even indexed elements. Therefore, by the induction hypothesis, after  $2k_r$  applications of  $T$ , we have a vector whose entries are all divisible by 2. By mathematical induction, for all natural  $n$ , a finite number of applications of  $T$  will transform any vector into a vector divisible by 2. As we concluded in the previous solution, a finite number of applications of  $T$  will transform any vector into the zero vector.

# The Cantor Set and Cantor Function

Naoki Sato

We begin by considering a problem that was given out at IMO training this past summer.

**Problem.** Let  $f : [0, 1] \rightarrow [0, 1]$  be a function satisfying the following three properties:

- (i)  $f$  is non-decreasing (that is;  $x < y \Rightarrow f(x) \leq f(y)$ ),
- (ii)  $f(x) = 1 - f(1 - x)$  for all  $x \in [0, 1]$ , and
- (iii)  $f(3x) = 2f(x)$  for all  $x \in [0, \frac{1}{3}]$ .

Evaluate  $f(1/7)$  and  $f(1/13)$ .

The reader at this point should stop and try to work out the values the problem asks for; this is a good way to get a feel for the dynamics of this function, which turns out to be a very special function, and well-known in analysis and chaos theory.

What is the first of many remarkable facts is that the three properties given above are enough to determine the value of  $f(x)$  for any point  $x \in [0, 1]$ , even though they do not seem to be – what if  $x$  is irrational?

We will come back to these questions later. We will first describe a seemingly unrelated concept, the Cantor set. There are several possible definitions, and this is perhaps the most straightforward.

Let  $A_0$  be the interval  $[0, 1]$ . From this, we remove the “middle-third”  $(\frac{1}{3}, \frac{2}{3})$ , and we obtain  $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ , which is the union of two intervals. We remove the middle-third from these two intervals, and obtain  $A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ , and so forth (see Figure 1). We call  $C$ , the set obtained in this limiting process, the Cantor set. Rigorously speaking,

$$A_n = [0, 1] \setminus \bigcup_{k=1}^n \bigcup_{j=1}^{3^{k-1}} \left( \frac{3j-2}{3^k}, \frac{3j-1}{3^k} \right),$$

$$C = [0, 1] \setminus \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{3^{k-1}} \left( \frac{3j-2}{3^k}, \frac{3j-1}{3^k} \right).$$

Note that  $A_n$  is the union of closed intervals, and that the sum of the lengths of these intervals is  $(\frac{2}{3})^n$  (since we remove one third at each step), which goes to 0 as  $n$  approaches infinity. In this sense, we have removed “most” of the interval  $[0,1]$  in obtaining  $C$ .



We mentioned that  $C$  has alternate definitions. Each  $x \in [0, 1]$  has a base 3 expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},$$

where  $a_i \in \{0, 1, 2\}$ . Then  $C$  is the set of all  $x \in [0, 1]$  such that  $a_i \neq 1$  for all  $i$ . In other words, each digit is a 0 or 2.

*Caution.* There are some cases where this is not exactly true — some numbers have two possible base 3 expansions, for example  $\frac{1}{3} = 0.1_3 = 0.0222\dots_3$ . Which one should we use?

The set of numbers which have  $a_1 = 1$  is precisely the interval  $(\frac{1}{3}, \frac{2}{3})$ , which is what we threw away from  $A_0$  to get  $A_1$ . Then, the set of numbers which have  $a_1 \neq 1$  and  $a_2 = 1$  is  $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$ , which is what we threw away from  $A_1$  to get  $A_2$ , and so on. Hence, the two constructions are the same.

The Cantor set has many uses. For example, although most of  $[0, 1]$  has been removed to obtain  $C$ , it can be shown that there is an onto map from  $C$  to  $[0, 1]$ ; that is, every element in  $[0, 1]$  is equal to the image of  $x$  for some  $x \in C$ . Not even  $\mathbb{Q}$ , the rationals, can make this claim. In the language of set theory,  $\mathbb{Q}$  is a countable set, and  $C$  and  $[0, 1]$  are uncountable sets, which are “bigger” (at this level, the size, or cardinality of sets are measured by the existence of 1–1, onto, or bijective maps between them).

Also, as the reader might have suspected, the Cantor set is also one of the most basic examples of a fractal, an object, roughly speaking, with self-similarity. In fact, the Cantor set is one of the most important fractals, despite its almost bland simplicity and lack of interesting detail. For example, certain Julia sets are modified Cantor sets. But how does all this relate to our original problem?

We give a formulation for  $f(x)$ . If  $x \in C$ , then  $a_i \in \{0, 2\}$  for all  $i$  in the base 3 expansion of  $x$ , and

$$f(0.a_1a_2a_3\dots_3) = 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\dots_2.$$

Note we have base 3 on the left and base 2 on the right. If  $x \notin C$ , then  $a_i = 1$  for some  $i$ . Choose the smallest such  $i$ . Then

$$f(0.a_1a_2a_3\dots a_i\dots_3) = 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\dots\frac{a_{i-1}}{2}a_i2.$$

For example,

$$f\left(\frac{4}{13}\right) = f(0.022022\dots_3) = 0.011011\dots_2 = \frac{3}{7},$$

$$f\left(\frac{38}{243}\right) = f(0.02102_3) = 0.011_2 = \frac{3}{8}.$$

We leave it to the reader to verify that these are the correct expressions, but we will show that the properties above are sufficient to determine  $f$ .

First, assume that  $x \in C$ , so  $a_i \in \{0, 2\}$  for all  $i$ . We proceed by induction on the number of digits of  $x$ . Let  $0.b_1b_2b_3\dots_2$  be the binary expansion of  $f(x)$ . Using the properties, we see that  $f(0) = 0$ ,  $f(1) = 1$ , and  $f(\frac{1}{3}) = f(\frac{2}{3}) = \frac{1}{2}$ , or  $f(0.0222\dots_3) = 0.0111\dots_2$ , and  $f(0.2_3) = 0.1_2$ . Hence, by expressing  $\frac{1}{2}$  in these two forms, and since  $f$  is non-decreasing, we see that if  $a_1 = 0$ , then  $b_1 = 0$ , and if  $a_1 = 2$ , then  $b_1 = 1$ , so  $b_1 = a_1/2$ .

Now assume  $b_i = a_i/2$  for all  $i$  from 1 up to some  $n$ . We must show that  $b_{n+1} = a_{n+1}/2$ . First, consider the case  $a_1 = 0$ . Then  $b_1 = 0$ , as shown above, and  $f(3x) = 2f(x)$ , or

$$f(0.a_2a_3\dots a_{n+1}\dots_3) = 0.b_2b_3\dots b_{n+1}\dots_2.$$

By the induction hypothesis,  $b_{n+1} = a_{n+1}/2$ . The case where  $a_1 = 2$  is left to the reader [Hint: This is where we must use  $f(x) = 1 - f(1 - x)$ ]. Therefore, by induction,  $b_i = a_i/2$  for all  $i$ .

Also, it can be seen that if  $x$  and  $y$  are the end-points of one of the intervals removed to obtain  $C$ , then  $f(x) = f(y)$ . Thus, again since  $f$  is non-decreasing,  $f$  on each missing interval is the common value at the end-points.

We also get another remarkable property as a bonus. Note that  $f$  is onto. Pick  $y \in [0, 1]$ , express  $y$  in base 2, and it should be obvious which  $x \in [0, 1]$  satisfies  $f(x) = y$ . We can even choose  $x \in C$ , so that  $f$ , when restricted to  $C$ , is an example (in fact, the standard example) of a map from  $C$  to  $[0, 1]$  that is onto. Moreover,  $f$  is non-decreasing, so the only possible discontinuities of  $f$  are jump discontinuities. But  $f$  is onto, so it cannot have any jump discontinuities, so  $f$  is continuous.



Figure 1.

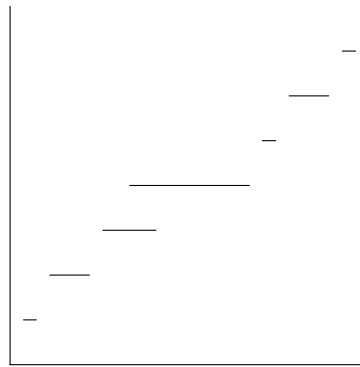


Figure 2.

When  $f$  is graphed, the features alluded to above become immediately apparent (see Figure 2). On the interval  $(\frac{1}{3}, \frac{2}{3})$ ,  $f$  is indeed the constant  $\frac{1}{2}$ ,

on  $(\frac{1}{9}, \frac{2}{9})$ , the constant  $\frac{1}{4}$ , and on  $(\frac{7}{9}, \frac{8}{9})$ , the constant  $\frac{3}{4}$ , and so on. As mentioned, this function  $f$ , sometimes called the Devil's Staircase, has some exceptional properties. By the observation just made,  $f'(x) = 0$  almost everywhere ("almost everywhere" does have a technical meaning, and it is related to the fact that most of  $[0, 1]$  is missing); in fact, the derivative is not defined precisely at points in the Cantor set. But  $f(0) = 0$  and  $f(1) = 1$ , so  $f$  must somehow climb up abruptly at points in the Cantor set, since it is flat almost everywhere.

So, a fairly innocuous problem on a functional equation turns out to have some large ramifications. We end with a few more miscellaneous facts.

Let  $T_1(x) = x/3$  and  $T_2(x) = (x + 2)/3$ . Pick a random point  $x \in [0, 1]$ , and recursively apply  $T_i$  to  $x$  (where  $i$  is randomly chosen at each step). Plot each  $x$ ; you can do this on your computer. What do you get?

Define  $T(x)$  on  $[0, 1]$  as follows:

$$T(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 1/2 \\ 3 - 3x & \text{if } 1/2 \leq x \leq 1 \end{cases}.$$

This function is sometimes called the tent function, for obvious reasons. Some points in the range of  $T$  are outside of  $[0, 1]$ ; which ones? These are bad points we wish to remove from the domain of  $T$ , because we wish to apply  $T$  an arbitrarily number of times to values in  $[0, 1]$ . For which  $x$  is it true that  $T^2(x) \in [0, 1]$ ? What about  $T^3$ ?

In fact, what is the set of all  $x$  such that  $T^k(x) \in [0, 1]$  for all  $k$ ?

You guessed it, it is the Cantor set. In terms of the base 3 expansion, how does  $T(x)$  relate to  $x$ ?

## Problems

1. Show that for all  $z \in [0, 2]$ , there exist  $x, y \in C$  such that  $x + y = z$ . Symbolically,  $C + C = [0, 2]$ .
2. Show that  $C$  is totally disconnected; that is, show that for all  $x, z \in C$ , there is a  $y \notin C$  such that  $x < y < z$ .



## J.I.R. McKnight Problems Contest 1981

1. If  $x$ ,  $y$ , and  $z$  are all between 0 and  $\frac{\pi}{2}$  inclusive, solve for  $x$ ,  $y$  and  $z$  if

$$\log_2(\sin x \sin y \sin z) = -\frac{3}{2}$$

$$\log_2\left(\frac{\sin^2 x \sin^3 y}{\sin z}\right) = -\frac{5}{2}$$

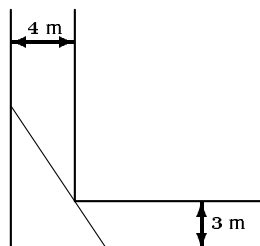
$$\log_2\left(\frac{\sin x \sin^4 y}{\sin^2 z}\right) = -3$$

2. A variable chord of an ellipse subtends a right angle at the centre. Show that the chord always touches a fixed circle.
3. (a) Obtain the equation of the tangent with slope  $m$  to the parabola whose equation is  $y^2 = 4px$ . Assume  $p > 0$ .
- (b) Obtain the equation of the tangent perpendicular to the tangent in (a).
- (c) Find the equation of the locus of the points of an intersection of pairs of perpendicular tangents to the parabola in (a).
4. Find the first four terms in ascending powers of  $x$  in the expansion as an infinite series of

$$\frac{1 - 2x}{1 + x - 2x^2}$$

and state the restrictions on  $x$ .

5. A tent has the shape of a cone and has a capacity of  $1000 \text{ m}^3$ . Find the radius of the base if the amount of canvas used is to be a minimum. (No canvas is used on the floor.)
6. Find the length of the longest ladder that can be carried horizontally around the corner of the corridor shown in the diagram.



# Mayhem Problems

The Mayhem Problems editors are:

**Richard Hoshino**    *Mayhem High School Problems Editor,*  
**Cyrus Hsia**        *Mayhem Advanced Problems Editor,*  
**David Savitt**       *Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from this issue be submitted by 1 April 1998, for publication in the issue 5 months ahead; that is, issue 6 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others.

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## High School Problems

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

**H223.** In each of the following alphametics, each letter in the addition represents a unique digit:

$$\begin{array}{r}
 1997 \\
 + \text{OLD} \\
 \hline
 \text{YEAR}
 \end{array}
 \qquad \text{and} \qquad
 \begin{array}{r}
 1998 \\
 + \text{OLD} \\
 \hline
 \text{YEAR}
 \end{array}$$

For each alphametic, find a solution, or prove that a solution does not exist.

**H224.** Let  $ABCD$  be a square. Construct equilateral triangles  $APB$ ,  $BQC$ ,  $CRD$ , and  $DSA$ , where  $P$ ,  $Q$ ,  $R$ , and  $S$  are points outside of the square.

- (a) Prove that  $PQRS$  is a square.
- (b) Determine the ratio  $\frac{PQ}{AB}$ . (See how many ways you can solve this!)

**H225.** Consider a row of five chairs, numbered 1, 2, 3, 4, and 5. You are originally sitting on 1. On each move, you must stand up and sit down on an adjacent chair. Make 19 moves, then take away chairs 1 and 5. Then make another 97 moves, with the three remaining chairs. No matter how the moves are made, you will always end up on chair 3. Why is this the case?

**H226.** The smallest multiple of 1998 that only consists of the digits 0 and 9 is 9990.

- (a) What is the smallest multiple of 1998 that only consists of the digits 0 and 3?
- (b) What is the smallest multiple of 1998 that only consists of the digits 0 and 1?

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## Advanced Problems

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada.  
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**A209.** Are there an infinite number of squares among the triangular numbers? Triangular numbers are numbers of the form  $T_n = n(n+1)/2$ .

**A210.** Let  $P$  be a point inside circle  $C$ . Find the locus of the centres of all circles  $\omega$  which pass through  $P$  and are tangent to  $C$ .

**A211.** Does there exist a convex polyhedron and a plane, not passing through any of its vertices, and intersecting more than  $\frac{2}{3}$  of all of the edges of the polyhedron?

(Polish Mathematical Olympiad, first round)

**A212.** Let  $A$  and  $B$  be real  $n \times n$  matrices such that  $A^2 + B^2 = AB$ . Prove that if  $AB - BA$  is an invertible matrix, then  $n$  is divisible by 3.

(International Competition for University Students in Mathematics)

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## Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University,  
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In this issue, we introduce David Savitt as the new Challenge Board editor. David is currently a graduate student at Harvard, and comes to us with much problem solving experience under his belt, including high ranking on the Putnam competition. It looks like he may be asking some very tough problems!

Ravi Vakil has recently graduated from Harvard University, and will be going on to Princeton and the IAS for post-doctoral work, and we wish him the best of luck.

**C75.** (a) Let  $n$  be an integer, and suppose  $a_1, a_2, a_3$ , and  $a_4$  are integers such that  $a_1 a_4 - a_2 a_3 \equiv 1 \pmod{n}$ . Show that there exist integers  $A_i$ ,  $1 \leq i \leq 4$ , such that each  $A_i \equiv a_i \pmod{n}$  and  $A_1 A_4 - A_2 A_3 = 1$ .

(b) Let  $SL(2, \mathbb{Z})$  denote the group of  $2 \times 2$  matrices with integer entries and determinant 1, and let  $\Gamma(n)$  denote the subgroup of  $SL(2, \mathbb{Z})$  of matrices which are congruent to the identity matrix modulo  $n$ . (By this we mean that all pairs of corresponding entries are congruent modulo  $n$ .)

What is the index of  $\Gamma(n)$  in  $SL(2, \mathbb{Z})$ ?

**C76.** Let  $X$  be any topological space. The  $n^{\text{th}}$  symmetric power of  $X$ , denoted  $X^{(n)}$ , is defined to be the quotient of the ordinary  $n$ -fold product  $X^n$  by the action of the symmetric group on  $n$  letters – that is, it is the space of unordered  $n$ -tuples of points of  $X$ . Show that the symmetric power  $\mathbb{C}^{(n)}$  is actually homeomorphic to the ordinary product  $\mathbb{C}^n$ .

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## From the files

For the benefit of those readers who are new to *MATHEMATICAL MAYHEM*, we present a few problems from back issues:

J14. [MAYHEM 1988: Vol 1, #1, 19]

In trapezoid  $ABCD$ , we have  $AB \parallel CD$  and  $|AB| = 2|CD|$ . Suppose that  $AC$  meets  $BD$  at  $X$ .

Find the ratio  $BX : XD$ .

S12. [MAYHEM 1988: Vol 1, #1, 20]

Find all functions with domain  $[0, \infty)$  such that

$$f(x) = \int_0^x f(t) dt.$$

U10. [MAYHEM 1988: Vol 1, #1, 21]

Prove that

$$\sum_{r=0}^n \binom{2n}{r} \binom{2n-2r}{n-r} = 4^n.$$


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