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SPECIAL CURVES IN FINSLER SPACE

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Abstract. As one point moves along a curve, if the Frenet vectors are carried to the center of the unit Finsler sphere FS^2 at this point then these vectors trace curves on the FS^2. These curves called as spherical images of the curve c. In this study, we investigate the Finslerian spherical images of any curve in Finsler space $\mathbb{F}^3$. Also, we obtain the Frenet-Serret formulas of these new curves in terms of Finsler invariants. Furthermore, these curves are exemplified using the Randers metric which is a special structure of the Finsler metric. Finally, these curves are illustrated on the FS^2.

1. Introduction

Finsler geometry began in 1918 with Paul Finsler. He obtained this geometry using minkowski norm instead of inner product. Thus, he was defining a more general metric including the Riemannian metric, which is called Finsler metric. Although the Finsler geometry shows a relatively slow development, it has now become comparable to modern differential geometry with its theorems and techniques. Finsler geometry is found many applications in thermodynamics, optics, ecology, evolution, biology etc. and has greatly improved clarity. For example, it is very important to find out what the concept of distance in terms of Finsler corresponds to in Euclidean sense. To find this answer, the Finsler metric is identified on the crystal optics and is found that the Finsler distance on this structure corresponds to the phase of an optical waveguide. Finsler geometry also emerges in field theory and particle physics. Studies related to this have been carried out by Kerbat *et al.* in 1990 on the evolution of cosmological models. In all of these cases, we can say that the Finsler metric is not an arbitrary mathematical structure, but rather physical, geometrical and biological interpretations, practices and experimentations. This gives us hope that the Finsler geometry sooner or later will be one of the important tools of physics, mathematics and nature, just like Riemann geometry. Thanks to its importance it has a great research field from geometry to biology, physics and also engineering and computer sciences.

The most informative works in these areas are the books given in [1, 4, 5, 6, 10]. Remizov [9] investigated singularities of geodesics flows in two-dimensional Finsler space. In ([11], [12]), Yıldırım *et al.* defined the helices in Finsler space and gave some characterizations for these curves. On the other hand, in [7] the authors

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obtained that the spherical images of the slant helices are spherical helices. In [8], Kula *et al.* characterized slant helices via the certain differential equations verified for each one of obtained spherical images in Euclidean $3-$space. In [2], Ali *et al.* gave the characterizations of slant helix in Minkowski $3-$space and they obtain that the spherical images of a slant helix is a helix.

In this article, we investigate the tangent, normal and binormal images of a curve in Finsler space $\mathbb{F}^3$, and give some characterizations for Frenet equations and curvature functions of these curves. Also, we obtain some related examples by using the Randers metric and these examples are illustrated on FS^2.

2. Basic Concepts and Arguments

In this section, we give the main geometric objects necessary for the study of Finsler geometry.

Definition 2.1. Let M' be non-empty open submanifold of TM such that $\pi(M') = M$ and $\theta(M) \cap M' = \emptyset$, where θ is a zero section of TM. It is given a smooth function $F : TM \to (0, \infty)$. Then suppose that any coordinate system $\{(U', \Phi'); x^i, y^i\}$ in M', the following conditions are fulfilled:
($\mathbf{F}_1$) F is positively homogeneous of degree one, that is, $F(\lambda X) = \lambda F(X)$ for all positive $\lambda > 0$, and $X \in TM$ or in the coordinate system, we have

$$F(x^1, ...x^m, \lambda y^1, ..., \lambda y^m) = \lambda F(x^1, ...x^m, y^1, ..., y^m),$$

for any $(x, y) \in \Phi'(U')$ and any $\lambda > 0$.
($\mathbf{F}_2$) For any nonzero $V \in T_xM$, the following quadratic form g_V on T_xM

$$g_V : T_xM \times T_xM \to \mathbb{R},$$

$$g_V(X, Y) = \langle X, Y\rangle_V := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(V + sX + tY)\right]_{|s=t=0}$$

is a positive definite quadratic form on $\mathbb{R}^m$ and using the canonical coordinates $(x, y) = (x^1, ..., x^n, y^1, ..., y^n)$ on TM, the coefficients of the metric g can be stated as follows

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y) \quad , \quad i, j \in \{1, ..., m\}.$$

The triple $\mathbb{F}^m = (M, M', F)$ with F satisfying ($\mathbf{F}_1$) and ($\mathbf{F}_2$) is a Finsler manifold and F is called the fundamental function of $\mathbb{F}^m$ [10].

Theorem 2.1. ***(Euler's Theorem)*** *A smooth function f on open positive conic subset D of $\mathbb{R}^m$ is a positively homogeneous of degree r if and only if it satisfies the condition*

$$y^i \frac{\partial f}{\partial y^i} = r f(y).$$

(i) $\dfrac{\partial F^2}{\partial y^i}$ *are positively homogeneous of degree one with respect to* $\left(y^1, y^2, ..., y^m\right)$.

(ii) $\dfrac{\partial F}{\partial y^i}$ *and* g_{ij} *are positively homogeneous of degree zero with respect to* $\left(y^1, y^2, ..., y^m\right)$.

2.1. Geometry of curves in Finsler manifold. Let $\mathbb{F}^{m+1}=(M,M',F)$ be a Finsler manifold and $\mathbb{F}^1=(c,c',F_1)$ be a $1-$dimensional Finsler submanifold of $\mathbb{F}^{m+1}$ where c is a smooth curve in M given locally by equations

$$x^i=x^i(s),\quad i\in\{1,2,...,m+1\},\quad s\in(a,b).$$

Denoted by (s,v) the coordinates on c', where s is the arc length parameter of the curve c. We consider (s,v) as local coordinates on c', we have

$$\frac{\partial}{\partial s}=\frac{dx^i}{ds}\frac{\partial}{\partial x^i}+v\frac{d^2x^i}{ds^2}\frac{\partial}{\partial y^i}$$

and

$$\frac{\partial}{\partial v}=\frac{dx^i}{ds}\frac{\partial}{\partial y^i}.$$

From here, it can be written

$$y^i(s,v)=v\frac{dx^i}{ds},\quad i\in\{0,...,m\}.$$

Moreover, $\left\{\frac{\partial}{\partial s},\frac{\partial}{\partial v}\right\}$ is a natural field of frames on c', where $\frac{\partial}{\partial v}$ is a unit Finsler vector field, that is $g\left(\frac{\partial}{\partial v},\frac{\partial}{\partial v}\right)=1$.

A Finsler vector field X on $\mathbb{F}^{m+1}$ along c' is projectable on c if locally at any point $(x(s),vx'(s))\in c'$ it is expressed as follows:

$$X(x(s),vx'(s))=X^i(s)\frac{\partial}{\partial y^i}\left(x\left(s\right),vx'(s)\right).$$

From the last equation, X on c' is defined a vector field X^* on c by the formula

$$X^*(x(s))=X^i(s)\frac{\partial}{\partial x^i}\left(x\left(s\right)\right).$$

So, the vector field $X^*(x(s))$ can be considered as the projection of the Finsler vector $X(x(s),vx'(s))$ on the tangent space TM of M at $x(s)\in c$ *(see for details* [5]).

Proposition 2.1. *The covariant derivatives of any projectible Finsler vector field X in the direction of $\frac{\partial}{\partial v}$ with respect to Cartan connection vanish identically on c', that is we have*

$$\left(\nabla_{\partial/\partial v}X\right)\left(x\left(s\right),vx'(s)\right)=0,\ \ s\in\left(-\varepsilon,\varepsilon\right).$$

In particular case, we get

$$\nabla_{\partial/\partial v}\frac{\partial}{\partial v}=0$$

which enables us to state that the **vertical covariant derivatives** along c with respect to Cartan connection do not provide any Frenet frame for c.

So, that the non-linear connection Hc' on $\mathbb{F}^1$ by the canonical non-linear connection GM' of $\mathbb{F}^{m+1}$ is locally spanned by $\frac{\partial}{\partial s}$. The Cartan connection is the best choice for studying the geometry of curves in a Finsler manifold. The Cartan derivative of the vector field $\dfrac{\partial}{\partial v}$ is obtained as folows:

$$\nabla^*_{\partial/\partial s}\frac{\partial}{\partial v}=\left(\frac{d^2x^k}{ds^2}+2G^k(s)\right)\frac{\partial}{\partial y^k}.$$

On the other hand, using $g(\frac{\partial}{\partial v},\frac{\partial}{\partial v})=1$ and taking into account that ∇^* is a metric linear connection we obtain

$$g\left(\nabla^*_{\partial/\partial s}\frac{\partial}{\partial v},\frac{\partial}{\partial v}\right)=0,$$

so,

$$\nabla^*_{\partial/\partial s}\frac{\partial}{\partial v}=\kappa_1 N_1$$

where N_1 is a unit Finsler vector field and

$$\kappa_1=F\left(\nabla^*_{\partial/\partial s}\frac{\partial}{\partial v}\right)$$

and using the local coordinates it is obtained that

$$\kappa_1=\left\{g_{ij}(s)\left(x''^i(s)+2G^i(s)\right)\left(x''^j(s)+2G^j(s)\right)\right\}^{1/2}$$

call it the **geodesic curvature function** of c and where

$$G^k=\frac{g^{kl}}{4}\left\{\frac{\partial^2F^2}{\partial s^2}y^m-\frac{\partial F^2}{\partial s}\right\}$$

and $g^{kl}=(g_{kl})^{-1}$.

If we take $m=2$, it is obtained the Frenet frame in Finsler space $\mathbb{F}^3$ [5].

Definition 2.2. The sphere of radius $r>0$ and with center in the origin in the Finsler space $\mathbb{F}^3$is defined by

$$FS^2:=\{X\in\mathbb{E}^3\,|\,F(X)=r\}.$$

Let $c=c(s)$ be a smooth curve in the 3−dimensional Finsler space $\mathbb{F}^3$ and s is an arc length parameter of the curve c. Denote by $\{T:=\dfrac{\partial}{\partial v},N,B\}$ the moving Frenet frame along the curve c in the Finsler space $\mathbb{F}^3$. Then, we have the derivations of the Frenet frame of the curve c are given by

$$\begin{aligned}\nabla^*_{\frac{\partial}{\partial s}}T&=\kappa N,\\ \nabla^*_{\frac{\partial}{\partial s}}N&=-\kappa T+\tau B,\\ \nabla^*_{\frac{\partial}{\partial s}}B&=-\tau N\end{aligned}$$

where N and B are called principal normal and binormal Finsler vector fields on the curve c, respectively. The principal curvature(or geodesic curvature) of the curve c is given by

$$\kappa(s)=\left\{g_{ij}(s)(\alpha''^i(s)+2G^i(s))(\alpha''^j(s)+2G^j(s))\right\}^{\frac{1}{2}}$$

and Finsler torsion of the curve c is

$$\tau(s)=-g\left(\nabla^*_{\frac{\partial}{\partial s}}N,B\right)(s)=-g_{ij}(s)B^i(s)\left\{\frac{\partial N^j}{\partial s}+N^k(s)S^j_k(s)\right\}$$

[5].

Corollary 2.1. *If* $\tau(s) = 0$ *for every* s *in the Finsler space* F^3 *then, the curve* c *lie on the plane of the* $\mathbb{R}^3$ [5].

3. Finslerian spherical images of the curve in $\mathbb{F}^3$

We can construct curves whose position vectors are the vector fields $T = \dfrac{\partial}{\partial v}$, N or B along the curve c. These curves lie on the unit Finsler sphere FS^2 and they are called Finslerian spherical images of the curve c. Now, we shall define step by step as follows:

Definition 3.1. The locus of points whose position vector is the vector $T = \dfrac{\partial}{\partial v}$ along to the curve c is called the tangent Finslerian spherical image c_T of the curve c.

Definition 3.2. The locus of points whose position vector is the vector N along to the curve c is called the normal Finslerian spherical image c_N of the curve c.

Definition 3.3. The locus of points whose position vector is the vector B along to the curve c is called the tangent Finslerian spherical image c_B of the curve c.

Let c_i be a spherical images of the curve c where i will be consider as a notion $T = \dfrac{\partial}{\partial v}$, N, B. In this section, we obtain the Frenet frames of the curves c_i in terms of the Frenet vector fields of the curve c in the Finsler space $\mathbb{F}^3$.

3.1. Tangent image of the curve c in Finsler space. Let $c := c(s)$ be a smooth curve in the 3−dimensional Finsler space $\mathbb{F}^3$ and s is an arclength parameter of the curve c. If we translate of the first (tangent) vector field of Frenet frame to the center O of the unit sphere FS^2, we obtain a spherical image $c_T(s_T)$, that is, $c_T(s_T) = \dfrac{\partial}{\partial v}$ where $\varphi : I \to I_T$, $s_T = \varphi(s)$ is a regular C^∞−function. One can Cartan differentiate of c_T with respect to s and taking into account that ∇^* is a metric linear connection, it can be calculated

$$\begin{aligned} c_T' &= \nabla^*_{\frac{\partial}{\partial s_T}} c_T \\ &= \nabla^*_{\frac{\partial s}{\partial s_T}\frac{\partial}{\partial s}} \frac{\partial}{\partial v} \\ &= \frac{\partial s}{\partial s_T} \nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} \\ &= \frac{\partial s}{\partial s_T} \kappa N. \end{aligned}$$

If we take a Finsler norm of the both side in the last equation, we obtain the tangent vector of the curve c_T as follows:

$$T_T = \frac{c_T'}{F(c_T')} = N$$

where $F(c_T') = \frac{\partial s}{\partial s_T}\kappa$ and since $\dfrac{\partial s}{\partial s_T} = \dfrac{1}{\kappa}$.

In order to determine the first curvature of c_T, we write

$$\begin{aligned}\nabla^*_{\frac{\partial}{\partial s_T}} T_T &= \nabla^*_{\frac{\partial}{\partial s_T}} N \\ &= \frac{1}{\kappa}\left(-\kappa T + \tau B\right).\end{aligned}$$

Hence, we may set that

$$\begin{aligned}\kappa_T &= F(\nabla^*_{\frac{\partial}{\partial s_T}} T_T) \\ &= \frac{\sqrt{\kappa^2+\tau^2}}{\kappa}.\end{aligned}$$

So, we find the normal vector field of the curve c_T

$$N_T = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa T + \tau B\right).$$

In order to obtain the third Finsler vector field from the Frenet frame of the curve c_T, we set

$$(T_T)_i = N_i = g_{ij} n^j \qquad \text{and} \qquad (N_T)_i = g_{ij} n^j_T.$$

Then we define

$$b^1 = \begin{vmatrix} n_2 & n_3 \\ (n_T)_2 & (n_T)_3 \end{vmatrix}, \quad b^2 = \begin{vmatrix} n_3 & n_1 \\ (n_T)_3 & (n_T)_1 \end{vmatrix}, \quad b^3 = \begin{vmatrix} n_1 & n_2 \\ (n_T)_1 & (n_T)_2 \end{vmatrix}, \quad b = g_{ij} b^i b^j.$$

This enables us to define the unit Finsler vector field

$$B_T = (b_T)^i \frac{\partial}{\partial y^i} \quad \text{where} \quad (b_T)^i = \frac{b^i}{\sqrt{b}}, \ i \in \{1,2,3\}.$$

The binormal vector field of the curve c_T is obtained

$$B_T = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau T + \kappa B\right).$$

From here, we have

$$\begin{aligned}\nabla^*_{\frac{\partial}{\partial s_T}} N_T &= \frac{1}{\kappa}\nabla^*_{\frac{\partial}{\partial s}}\left(\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa T + \tau B\right)\right) \\ &= \frac{1}{\kappa}\left[\left(\nabla^*_{\frac{\partial}{\partial s}}\frac{1}{\sqrt{\kappa^2+\tau^2}}\right)\left(-\kappa T + \tau B\right) + \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\nabla^*_{\frac{\partial}{\partial s}}\left(-\kappa T\right) + \nabla^*_{\frac{\partial}{\partial s}}\left(\tau B\right)\right)\right] \\ &= \frac{1}{\kappa\left(\kappa^2+\tau^2\right)^{3/2}}\left\{\tau\left(\kappa\tau' - \tau\kappa'\right)T - \left(\kappa^2+\tau^2\right)^2 N + \kappa\left(\kappa\tau' - \tau\kappa'\right)B\right\}\end{aligned}$$

and torsion function of this curve is given by

$$\begin{aligned}\tau_T(s_T) &= g(\nabla^*_{\frac{\partial}{\partial s_T}} N_T, B_T) \\ &= \left(\frac{\kappa}{\kappa^2+\tau^2}\right)\left(\nabla^*_{\frac{\partial}{\partial s}}\left(\frac{\tau}{\kappa}\right)\right).\end{aligned}$$

Corollary 3.1. *If the curve c is a helix (i.e. the ratio of Finsler curvatures of the curve is a constant as defined in* [11]*), then the tangent Finslerian spherical image curve c_T is a Finsler circle.*

3.2. Normal image of the curve c in Finsler space. Let $c = c(s)$ be a smooth curve in the $3-$dimensional Finsler space $\mathbb{F}^3$ with arclength parameter s. If we translate of the second (normal) vector field of Frenet frame to the center O of the unit Finsler sphere FS^2, we obtain a spherical image $c_N(s_N)$, that is, $c_N(s_N) = N(s)$ where $\varphi : I \to I_N$, $s_N = \varphi(s)$ is a regular $C^\infty-$function. If we differentiate the curve c_N with respect to s_N and taking into account that ∇^* is a metric linear connection, then we get

$$\begin{aligned} c_N' &= \nabla^*_{\frac{\partial}{\partial s_N}} N \\ &= \frac{\partial s}{\partial s_N} \nabla^*_{\frac{\partial}{\partial s}} N \\ &= \frac{\partial s}{\partial s_N} \left(-\kappa T + \tau B\right). \end{aligned}$$

If we take a Finsler norm of the both side in the last equation, we have the tangent vector of the curve c_N as follows:

$$T_N = \frac{c_N'}{F(c_N')} = \frac{1}{\sqrt{\kappa^2+\tau^2}} \left(-\kappa T + \tau B\right).$$

where $\dfrac{\partial s}{\partial s_N} = \dfrac{1}{\sqrt{\kappa^2+\tau^2}}$. This gives

$$\nabla^*_{\frac{\partial}{\partial s_N}} T_N = \frac{1}{\sqrt{\kappa^2+\tau^2}} \left\{ \frac{\tau\left(\kappa\tau' - \tau\kappa'\right)}{\left(\kappa^2+\tau^2\right)^{3/2}} T - \sqrt{\kappa^2+\tau^2} N + \frac{\kappa\left(\kappa\tau' - \tau\kappa'\right)}{\left(\kappa^2+\tau^2\right)^{3/2}} B \right\},$$

$$\begin{aligned} \kappa_N &= F(\nabla^*_{\frac{\partial}{\partial s_N}} T_N) \\ &= \frac{\sqrt{\left(\kappa\tau' - \tau\kappa'\right)^2 + \left(\kappa^2+\tau^2\right)^3}}{\left(\kappa^2+\tau^2\right)^{3/2}}. \end{aligned}$$

Then, we find the normal vector field of the curve c_N

$$N_N = \frac{\left(\kappa^2+\tau^2\right)}{\sqrt{\left(\kappa\tau' - \tau\kappa'\right)^2 + \left(\kappa^2+\tau^2\right)^3}} \left\{ \frac{\tau\left(\kappa\tau' - \tau\kappa'\right)}{\left(\kappa^2+\tau^2\right)^{3/2}} T - \sqrt{\kappa^2+\tau^2} N + \frac{\kappa\left(\kappa\tau' - \tau\kappa'\right)}{\left(\kappa^2+\tau^2\right)^{3/2}} B \right\}.$$

The third Finsler vector field is obtained by

$$B_N = \frac{\left(\kappa^2+\tau^2\right)}{\sqrt{\left(\kappa\tau' - \tau\kappa'\right)^2 + \left(\kappa^2+\tau^2\right)^3}} \left\{ -\tau T + \frac{\left(\kappa\tau' - \tau\kappa'\right)}{\sqrt{\kappa^2+\tau^2}} N - \kappa B \right\}.$$

Finally, the torsion function of the curve c_N is calculated

$$\tau_N = \frac{A}{\left(\kappa^2+\tau^2\right)^{3/2}} \left\{ \begin{array}{c} -C\kappa^2\left(\kappa^2+\tau^2\right)\nabla^*_{\frac{\partial}{\partial s}} A - AC\kappa^2\left(\kappa\kappa' + \tau\tau'\right) \\ -\left(\kappa^2+\tau^2\right)^{3/2}\left(C\nabla^*_{\frac{\partial}{\partial s}} A - A\nabla^*_{\frac{\partial}{\partial s}} C\right) \end{array} \right\}$$

where $A = \frac{\left(\kappa^2+\tau^2\right)}{\sqrt{\left(\kappa\tau' - \tau\kappa'\right)^2 + \left(\kappa^2+\tau^2\right)^3}}$ and $C = \frac{\left(\kappa\tau' - \tau\kappa'\right)}{\sqrt{\kappa^2+\tau^2}}$.

3.3. Binormal image of the curve c in Finsler space. Let $c = c(s)$ be a smooth curve in the Finsler space $\mathbb{F}^3$ with arclength parameter s. If we translate of the binormal vector field of Frenet frame to the center O of the unit Finsler sphere FS^2, we obtain a spherical image $c_B(s_B)$, that is, $c_B(s_B) = B$ where $\varphi : I \to I_B$, $s_B = \varphi(s)$ is a regular $C^\infty-$function. One can Cartan differentiate of c_B with respect to s, taking into account that ∇^* is a metric linear connection and using the equation $\dfrac{\partial s}{\partial s_T} = \dfrac{1}{\tau}$ it can be calculated

$$T_B = -N.$$

Then Frenet apparatus of the curve c_B are similarly obtained as follows:

$$\begin{aligned} N_B &= \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\kappa T - \tau B\right), \\ B_B &= \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau T + \kappa B\right), \end{aligned}$$

$$\kappa_B = \frac{\sqrt{\kappa^2+\tau^2}}{\tau} \quad \text{and} \quad \tau_B = \left(\frac{\tau}{\kappa^2+\tau^2}\right)\left(\nabla^*_{\frac{\partial}{\partial s}}\left(\frac{\kappa}{\tau}\right)\right).$$

Corollary 3.2. *If the curve c is a helix as defined in* [11]*, then the curve c_B is a Finsler circle.*

4. Example

In this section, we give an example of a space curve c with arc length parameter s in Finsler space $\mathbb{F}^3$ and calculate its Frenet apparatus using the Randers metric which is defined as special Finsler metric was the first introduce by Randers in 1941. Then, we plot its figure and their tangent, normal, binormal Finslerian spherical images c_T, c_N, c_B by using Mathematica program, respectively.

Let $\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}$ be an Euclidean norm on a vector space $\mathbb{R}^n$ and $\beta(x) = b_i(x)y^i$ be a linear form on $\mathbb{R}^n$, then Randers metric is defined as follows:

$$F(x,y) := \alpha\left(x,y\right) + \beta\left(x,y\right). \tag{4.1}$$

In [4], authors obtain

$$g_{ij} = \frac{F}{\alpha}\left\{a_{ij} - \frac{y^i}{\alpha}\frac{y^j}{\alpha} + \frac{\alpha}{F}\left(b_i + \frac{y_i}{\alpha}\right)\left(b_j + \frac{y_j}{\alpha}\right)\right\}$$

where $y_i := a_{ij}y^j$. Since the bilinear form (g_{ij}) is positive definite, then the length of β is less then 1, i.e.,

$$\|\beta\|_\alpha := \sqrt{a^{ij}b_i b_j} < 1$$

where $\left(a^{ij}\right) := \left(a_{ij}\right)^{-1}$. A Minkowski norm in the form (4.1) is called the Randers norm ([4]).

Example 4.1. We consider a curve c with arc length parameter s in Finsler space $\mathbb{F}^3$ is defined by

$$c\left(s\right) = \left(\frac{2bs + \cos^2 s}{2(-1+b^2)}, \frac{2s + \sin 2s}{4\sqrt{1-b^2}}, \frac{-\cos s}{\sqrt{1-b^2}}\right)$$

where $b \in (0,1)$ is a constant real number. The curve c is plotted in Fig. 1.

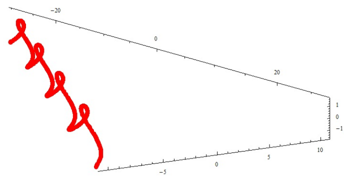

Figure 1. The curve c for $b = 0.8$.

The tangent vector along the curve c is given by

$$T(s) = \left(\frac{\cos s \sin s - b}{1 - b^2}, \frac{\cos^2 s}{\sqrt{1 - b^2}}, \frac{\sin s}{\sqrt{1 - b^2}} \right)$$

and the graph of the curve $c_T(s_T) = T(s)$ is included in Fig.2.

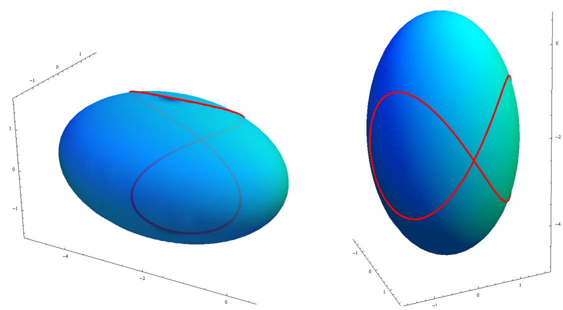

Figure 2. The tangent image c_T of the curve c on the Finsler sphere for $b = 0.8$.

The normal vector field along the curve c is obtained by

$$N(s) = \frac{1}{\sqrt{\cos^2 s + -2b^2 + 1 + b\cos 2s}} \left(\cos 2s, -\sqrt{1 - b^2} \sin 2s, \sqrt{1 - b^2} \cos s \right)$$

The graph of the curve $c_N(s_N) = N(s)$ can be deserved in Fig.3.

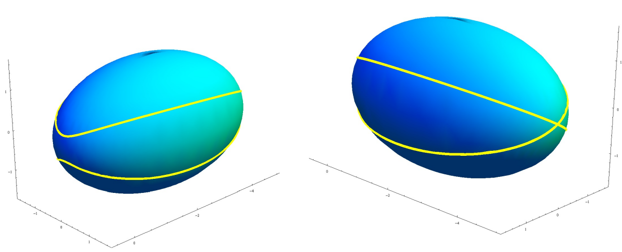

Figure 3. The normal image c_N of the curve c on the FS^2 for $b = 0.8$.

From here, we obtain the binormal vector field of the curve c as follows:

$$B(s)=\frac{1}{A}\left(\cos^3 s+\sin s\sin 2s,\ \frac{-b\cos s-\cos^2 s\sin s+\cos 2s\sin s}{\sqrt{1-b^2}},\ \frac{b\sin 2s-\cos^2 s\cos 2s-\frac{1}{2}\sin^2 2s}{\sqrt{1-b^2}}\right)$$

where

$$A\ :=\sqrt{\begin{array}{c}\left(\cos^3 s+\sin s\sin 2s\right)^2+\frac{1}{1-b^2}\left(-b\cos s-\cos^2 s\sin s+\cos 2s\sin s\right)^2\\ +\frac{1}{1-b^2}\left(b\sin 2s-\cos^2 s\cos 2s-\frac{1}{2}\sin^2 2s\right)^2\end{array}}$$
$$+b\left(\cos^3 s+\sin s\sin 2s\right).$$

Also, the graph of the curve $c_B(s_B)=B(s)$ can be found in Fig. 4.

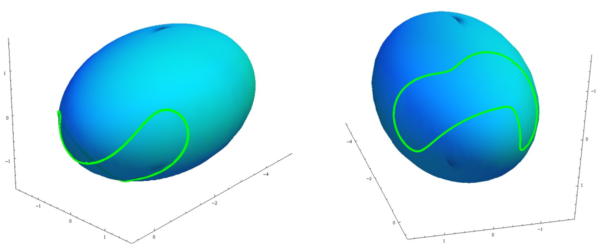

Figure 4. The binormal Finslerian spherical image c_B of the curve c $(b=0.8)$.

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